

TMA4215 Numerical Mathematics

Autumn 2010

Exercise 4

Task 1

- a) Use divided differences and Newton's interpolation formula to find the interpolating polynomial of lowest possible degree for the points in the table:

$$\begin{array}{c|ccc} x_i & 0 & 2 & 3 \\ \hline y_i & 1 & 2 & 4 \end{array}.$$

- b) Add the point $(1, 0)$ to the table in **a)**. What is the new interpolating polynomial?

Task 2

In this task, you are to approximate $\sin(x)$ on the interval $[0, \pi]$ using polynomial interpolation.

- a) Choose 4 equidistant nodes on the interval. Find the polynomial and an upper bound for the error $|\sin(x) - p_3(x)|$.
- b) Repeat **a)**, but use Chebyshev nodes instead (see hint). (You do not have to calculate the polynomial, but show the interpolation points.)
- c) In the two cases (equidistant and Chebyshev nodes), find an expression for an upper error bound, expressed by n (the polynomial degree). Plot the error bound as a function of n and compare the two cases.

Hint: The Chebyshev nodes are defined on the interval $[-1, 1]$. To move them over to another interval $[a, b]$, the change of variables

$$x = \frac{a+b}{2} + \frac{b-a}{2}t, \quad t \in [-1, 1]$$

is used. The error formula is adjusted in the same way.

Task 3

Write two MATLAB functions, one that calculates the table of divided differences based on a given dataset, and one that calculates the value of the interpolating polynomial in given points, based on this table. E.g.

```
function tab = divdiff(x, y)
```

and

```
function y = pval(tab, t)
```

where \mathbf{t} may be a vector.

Test the functions on the dataset in Task 1.

Task 4

Given Runge's function $f(x) = 1/(1 + 25x^2)$, $x \in [-1, 1]$. Find and plot the polynomial interpolating f in equidistant nodes. Use $n = 6, 11$ and 21 . Use the `polyfit` function in MATLAB. This function generates a polynomial of degree n interpolating given data. If we use $n + 1$ data points, and the degree of the polynomial is n , the obtained polynomial is the interpolation polynomial. Use the function `polyval` to evaluate the obtained (interpolation) polynomial on a desired range of values, and then plot the results.

Repeat the experiment with Chebyshev nodes and comment the result.

Task 5

Let the distance h between nodes be such that $x_i = a + ih$, $i = 0, 1, 2, \dots$. Let $f_i = f(x_i)$, $i = 0, 1, 2, \dots$

On such a sequence $\{f_i\}_{i=0}^{\infty}$, we can define a *forward difference* recursively by

$$\Delta^0 f_0 = f_0, \quad \Delta f_0 = f_1 - f_0, \quad \Delta^k f_0 = \Delta(\Delta^{k-1} f_0) = \Delta^{k-1} f_1 - \Delta^{k-1} f_0, \quad k = 1, 2, \dots$$

We get

$$\Delta^2 f_0 = f_2 - 2f_1 + f_0, \quad \Delta^3 f_0 = f_3 - 3f_2 + 3f_1 - f_0, \quad \text{and so on.}$$

Let $x = x_0 + sh$, where $s \in \mathbb{R}$. The task is about showing that the polynomial interpolating f in the nodes x_i , $i = 0, 1, \dots, n$ can be written

$$p_n(x) = p_n(x_0 + sh) = f_0 + \sum_{k=1}^n \binom{s}{k} \Delta^k f_0 \tag{1}$$

where

$$\binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k!}.$$

a) Show by induction:

$$f[x_0, x_1, \dots, x_k] = \frac{1}{k!h^k} \Delta^k f_0.$$

b) Show that

$$\prod_{i=0}^{k-1} (x - x_i) = k!h^k \binom{s}{k}, \quad k \geq 1.$$

c) Use the results from **a)** and **b)** to prove Newton's forward difference formula (1).

d) Apply the formula to the dataset in Task **1b)**.

Comment: Equivalently, it is possible to show *Newton's backward difference formula*. Backward differences on the sequence $\{f_n\}_{n=0}^{\infty}$ are defined by

$$\nabla^0 f_n = f_n, \quad \nabla f_n = f_n - f_{n-1}, \quad \nabla^k f_n = \nabla^{k-1} f_n - \nabla^{k-1} f_{n-1}, \quad k = 1, 2, \dots$$

Newton's backward difference formula is given by

$$p_n(x) = p_n(x_n + sh) = f_n + \sum_{k=1}^n (-1)^k \binom{-s}{k} \nabla^k f_n.$$

Some relevant suggested exam problems:

Those of you that want some more problems/examples can take a look at the following exam problems:

December 2007, problem 1.

December 2006, problem 1.

August 2006, problem 2.

August 2005, problem 2.