TMA4215 Numerical Mathematics

Autumn 2010

Exercise 6

Task 1

Show that the second column in the Romberg array is the same as what you get by using the composite Simpson's rule.

Task 2

Write a MATLAB program that calculates the Romberg array for a given integral. Stop when $|R(n,n) - R(n-1,n-1)| \leq tol$, where tol is the tolerance supplied by the user. Test the program on the integrals

$$\int_0^1 \sin(x)\,\mathrm{d}x, \qquad \int_0^1 \sqrt{x}\sin(x)\,\mathrm{d}x.$$

Comment the results. (See the exam 2008, problem 2).

Task 3

Construct an adaptive trapezoid algorithm. Apply the algorithm to find the value of the *Fresnel integral*

$$S(x) = \int_0^x \sin(t^2)\,\mathrm{d}t.$$

for $x = 0.8$. Use $tol = 2 \cdot 10^{-3}$.

Task 4

a) Find an approximation to the integral

$$\int_{-1}^1 \frac{\mathrm{e}^x}{\sqrt{1-x^2}}\,\mathrm{d}x$$

by using Gaussian quadrature with $n = 1$ (two nodes). Use the Gaussian quadrature based on the Legendre polynomials.

b) Find a Gaussian quadrature of the form

$$\int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}}\,\mathrm{d}x \approx A_1 f(x_1) + A_2 f(x_2)$$

and use this to calculate the integral in **a)**. Compare with the exact solution.

c) Find an upper limit for the error in **b)**.

Task 5

Find the first three Laguerre polynomials, i.e. polynomials that are orthogonal with respect to the inner product

$$\langle p, q\rangle = \int_0^\infty \mathrm{e}^{-x} p(x)q(x)\,\mathrm{d}x.$$

Task 6

Show that the polynomials defined by

$$\Phi_k(x) = \frac{1}{2^k\,k!}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[(x^2-1)^k\right]$$

are orthogonal with respect to the inner product

$$\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,\mathrm{d}x.$$

Hint: Note that the j-th derivative of $(x^2-1)^k$ is divisible by (x^2-1) if $j<k$. Use partial integration repeatedly.

Relevant exam problems:

- December 2008, problem 3,
- December 2007, problem 3,
- December 2006, problem 2,
- August 2006, problem 3.