TMA4215 Numerical Mathematics

Autumn 2010

Solution 7

Task 1

a) We have

$$
\begin{aligned}
&\text{Exact solution:} \qquad & y(t_{n+1}) &= y(t_n) + h\Phi\big(t_n, y(t_n); h\big) + d_{n+1}, \\
&\text{Numerical solution:} \qquad & y_{n+1} &= y_n + h\Phi(t_n, y_n; h).
\end{aligned}
$$

Take the difference between these, and use the fact that the global error in step n is given by $e_n = y(t_n) - y_n$:

$$e_{n+1} = e_n + h\Big(\Phi\big(t_n, y(t_n); h\big) - \Phi(t_n, y_n; h)\Big) + d_{n+1}.$$

Take the norm of both sides and apply the triangle inequality for norms:

$$
\begin{aligned}
\|e_{n+1}\| &= \Big\|e_n + h\Big(\Phi\big(t_n, y(t_n); h\big) - \Phi(t_n, y_n; h)\Big) + d_{n+1}\Big\| \\
&\le \|e_n\| + h\big\|\Phi\big(t_n, y(t_n); h\big) - \Phi(t_n, y_n; h)\big\| + \|d_{n+1}\|.
\end{aligned}
$$

Use the assumptions given in the text:

$$\|e_{n+1}\| \le \|e_n\| + hM\|y(t_n) - y_n\| + Dh^{p+1} = (1 + hM)\|e_n\| + Dh^{p+1}.$$

Assuming that $y(t_0) = y_0$, we now have

$$
\begin{aligned}
\|e_1\| &\le Dh^{p+1}, \\
\|e_2\| &\le (1 + hM)Dh^{p+1} + Dh^{p+1}, \\
&\vdots \\
\|e_n\| &\le Dh^{p+1}\sum_{i=0}^{n-1}(1 + hM)^i, \\
&\vdots
\end{aligned}
$$

so that

$$\|e_N\| \le Dh^{p+1}\sum_{i=0}^{N-1}(1 + hM)^i = \frac{(1 + hM)^N - 1}{1 + hM - 1}Dh^{p+1} = \frac{(1 + hM)^N - 1}{M}Dh^p.$$

Utilise the fact that $\mathrm{e}^x \ge 1 + x$ for $x > 0$, so that

$$\|e_N\| \le \frac{\mathrm{e}^{MhN} - 1}{M}Dh^p = \frac{\mathrm{e}^{M(t_{\text{end}} - t_0)} - 1}{M}Dh^p = Ch^p.$$

This argument does not depend on the choice of norm.

b) We know that f satisfies a Lipschitz condition in y, i.e.

$$\|f(t, y) - f(t, \tilde{y})\| \leq L\|y - \tilde{y}\|.$$

The increment function $\Phi(t_n, y_n; h)$ for the given method is

$$\Phi(t_n, y_n; h) = b_1k_1 + b_2k_2.$$

Let us calculate this for two different starting values y_n and $\tilde{y}_n$.

$$\begin{aligned} k_1 &= f(t_n, y_n), & \tilde{k}_1 &= f(t_n, \tilde{y}_n), \\ k_2 &= f(t_n + c_2h, y_n + hc_2k_1), & \tilde{k}_2 &= f(t_n + c_2h, \tilde{y}_n + hc_2\tilde{k}_1). \end{aligned}$$

Thus,

$$\Phi(t_n, y_n; h) - \Phi(t_n, \tilde{y}_n; h) = b_1(k_1 - \tilde{k}_1) + b_2(k_2 - \tilde{k}_2).$$

Take the norm of both sides and use the properties of norms as well as Lipschitz continuity for f:

$$\begin{aligned} \|\Phi(t_n, y_n; h) - \Phi(t_n, \tilde{y}_n; h))\| &= \|b_1(k_1 - \tilde{k}_1) + b_2(k_2 - \tilde{k}_2)\| \\ &\leq |b_1|\|k_1 - \tilde{k}_1\| + |b_2|\|k_2 - \tilde{k}_2\| \\ &\leq |b_1|L\|y_n - \tilde{y_n}\| + |b_2|L\|y_n + hc_2k_1 - \tilde{y}_n - hc_2\tilde{k}_1\| \\ &\leq |b_1|L\|y_n - \tilde{y_n}\| + |b_2|L(\|y_n - \tilde{y}_n\| + h|c_2|L\|y_n - \tilde{y}_n\|) \end{aligned}$$

Using that $h \leq h_{\max}$, we have now shown that

$$\|\Phi(t_n, y_n; h) - \Phi(t_n, \tilde{y}_n; h))\| \leq M\|y_n - \tilde{y}_n\|$$

with

$$M = L(|b_1| + |b_2|) + h_{\max}L^2|b_2c_2|.$$

Task 2

a) We can write (4) from the text as a system of two first order differential equations by introducing the variable $v = u'$. Then

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} v \\ A\cos\omega t + u(1-u^2) - kv \end{pmatrix}.$$

b) By writing

$$\mathbf{u} = (u, v)^{\mathrm{T}}$$

and

$$\mathbf{f}(t, \mathbf{u}) = \big(v, A\cos\omega t + u(1-u^2) - kv\big)^{\mathrm{T}}$$

where superscript T means taking the transpose, we can write improved Euler for this problem as

$$\mathbf{u}_{m+1} = \mathbf{u}_m + \frac{h}{2}\Big(\mathbf{f}(t_m, \mathbf{u}_m) + \mathbf{f}\big(t_{m+1}, \mathbf{u}_m + h\mathbf{f}(t_m, \mathbf{u}_m)\big)\Big).$$

With the given starting values, we have

$$\mathbf{f}(t_0, \mathbf{u}_0) = \Big(v(0), A\cos(\omega\cdot 0) + u(0)\big(1 - u(0)^2\big) - kv(0)\Big)^{\mathrm{T}} = (0, A)^{\mathrm{T}},$$

and

$$\mathbf{f}\big(t_1, \mathbf{u}_0 + h\mathbf{f}(t_0, \mathbf{u}_0)\big) = \mathbf{f}\big(h, (0 + h\cdot 0, 0 + hA)^{\mathrm{T}}\big) = A(h, \cos\omega h - kh)^{\mathrm{T}}.$$

Thus,

$$\begin{aligned} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + A\frac{h}{2}\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} h \\ \cos\omega h - kh \end{pmatrix}\right] \\ &= \frac{Ah}{2}\begin{pmatrix} h \\ 1 + \cos\omega h - kh \end{pmatrix}, \end{aligned}$$

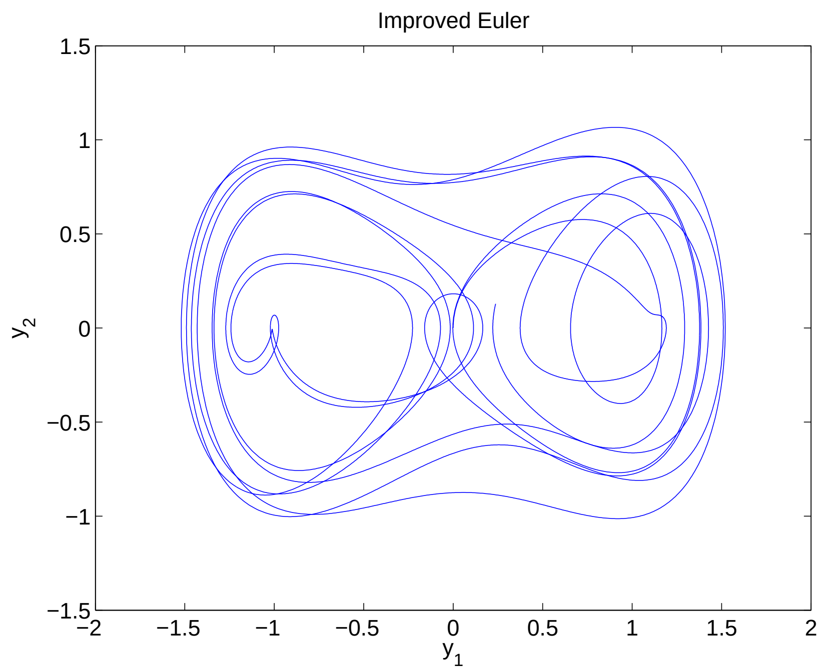
which becomes

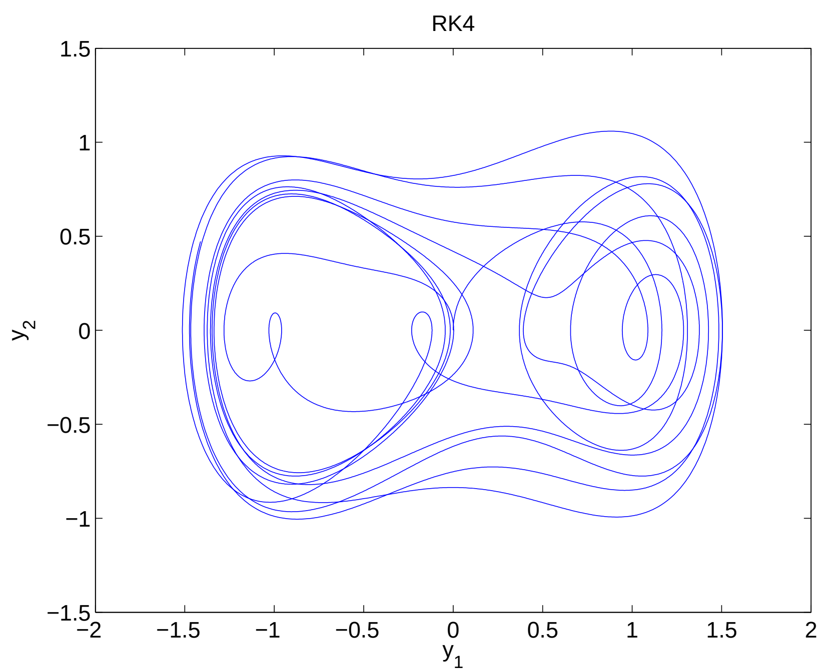
$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \approx \begin{pmatrix} 0{,}002 \\ 0{,}0394 \end{pmatrix}$$

for the given values.

For **c)**, **d)** and **e)**, see the supplied MATLAB files and try for yourself.

The result using improved Euler and RK4 are given in the figures below. Here we have used the parameters of **2b)** with $h = 0.01$ and integrated from 0 to 100.


Improved Euler
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
y_1
y_2


RK4
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
y_1
y_2

Task 3

a) The eight order conditions for fourth order Runge–Kutta methods are:

$$\sum_i b_i = 1 \tag{1}$$

$$\sum_i b_i c_i = \frac{1}{2} \tag{2}$$

$$\sum_i b_i c_i^2 = \frac{1}{3} \tag{3}$$

$$\sum_{i,j} b_i a_{ij} c_j = \frac{1}{6} \tag{4}$$

$$\sum_i b_i c_i^3 = \frac{1}{4} \tag{5}$$

$$\sum_{i,j} b_i c_i a_{ij} c_j = \frac{1}{8} \tag{6}$$

$$\sum_{i,j} b_i a_{ij} c_j^2 = \frac{1}{12} \tag{7}$$

$$\sum_{i,j,k} b_i a_{ij} a_{jk} c_k = \frac{1}{24} \tag{8}$$

To check that the given method satisfies the order conditions, we simply insert the values from the Butcher tableau. E.g. for (7), we get

$$\begin{aligned}\sum_{i,j} b_i a_{ij} c_j^2 &= b_3 a_{32} c_2^2 + b_4 (a_{42} c_2^2 + a_{43} c_3^2) \\ &= \frac{1}{3}\frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{6}\left(0 \cdot \left(\frac{1}{2}\right)^2 + 1 \cdot \left(\frac{1}{2}\right)^2\right) = \frac{1}{12}.\end{aligned}$$

b) We want to find a set of weights $\hat{b}_i$ so that the conditions (1)–(4) are satisfied. This means that

$$\begin{aligned}\hat{b}_1 + \hat{b}_2 + \hat{b}_3 + \hat{b}_4 &= 1, \\ \frac{1}{2}\hat{b}_2 + \frac{1}{2}\hat{b}_3 + \hat{b}_4 &= \frac{1}{2}, \\ \frac{1}{4}\hat{b}_2 + \frac{1}{4}\hat{b}_3 + \hat{b}_4 &= \frac{1}{3}, \\ \frac{1}{4}\hat{b}_3 + \frac{1}{2}\hat{b}_4 &= \frac{1}{6}.\end{aligned}$$

This system has a unique solution, namely the weights of the original Kutta's method, $\hat{b}_1 = \hat{b}_4 = \frac{1}{6}$ and $\hat{b}_2 = \hat{b}_3 = \frac{1}{3}$. Thus, we can not use this solution for comparison.

Task 4

a) The equation $y' = y$, $y(0)$ has solution $y(t) = \mathrm{e}^t y_0$. After one step, the solution is

$$y(h) = \mathrm{e}^h y_0 = \left(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots\right) y_0 = y_0 \sum_{q=0}^{\infty} \frac{h^p}{p!}.$$

For this problem, an explicit method with s stages is given by

$$\begin{aligned}
k_1 &= y_0 \\
k_2 &= y_0 + h a_{21} k_1 = (1 + h a_{21}) y_0 \\
k_3 &= y_0 + h a_{31} k_1 + h a_{32} k_2 = \big(1 + h(a_{31} + a_{32}) + h^2 a_{32} a_{21}\big) y_0 \\
&\vdots \\
k_s &= y_0 + h \sum_{j=1}^{s-1} a_{sj} k_j \\
y_1 &= y_0 + h \sum_{i=1}^{s} b_i k_i.
\end{aligned}$$

We see that k_i is a polynomial of degree $i - 1$ in h. This means that y_1 is a polynomial of degree s. Thus, the method can maximally be of order s.

Comment: There are explicit Runge–Kutta methods of order s for $s \leq 4$, but to find such a method of order 5, we need $s \geq 6$.

b) The 4 current order conditions for methods of order 3 with 3 stages are:

$$\begin{aligned}
b_1 + b_2 + b_3 &= 1, \\
b_2 c_2 + b_3 c_3 &= \frac{1}{2}, \\
b_2 c_2^2 + b_3 c_3^2 &= \frac{1}{3}, \\
b_3 a_{32} c_2 &= \frac{1}{6}.
\end{aligned}$$

This system is linear for b_1, b_2, b_3. We can neglect the first equation and b_1, and set up a system for b_2, b_3:

$$\begin{pmatrix} c_2 & c_3 \\ c_2^2 & c_3^2 \\ 0 & a_{32} c_2 \end{pmatrix} \begin{pmatrix} b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/3 \\ 1/6 \end{pmatrix}.$$

The right-hand side must be in the column space of the matrix for the system to have a solution. This implies that the 3×3 matrix we get by inserting the right-hand side as a third column must be singular, i.e. have a determinant of zero. The condition becomes

$$\begin{vmatrix} c_2 & c_3 & 1/2 \\ c_2^2 & c_3^2 & 1/3 \\ 0 & a_{32} c_2 & 1/6 \end{vmatrix} = 0 \qquad \Longrightarrow \qquad c_2 (3 a_{32} c_2^2 - 2 a_{32} c_2 - c_2 c_3 + c_3^2) = 0.$$

We see that $c_2 = 0$ is impossible since this would make the first column zero, and the second is not proportional with the third. We could also see this from the impossibility of satisfying the last order condition. The conclusion is the necessary and sufficient condition

$$3a_{32}c_2^2 - 2a_{32}c_2 - c_2c_3 + c_3^2 = 0.$$

c) We insert $a_{32} = c_3$ in the relation above and get the condition

$$c_3\big(c_3 - 3c_2(1-c_2)\big) = 0.$$

We can not have $c_3 = 0$ since then $a_{32} = 0$ and the final order condition would be impossible to satisfy. Instead we must demand

$$c_3 = 3c_2(1-c_2).$$

Note that all values of c_2, except $c_2 = 0, c_2 = 1$, are allowed. For all other values of c_2, the method is characterised by the free parameter c_2 as follows:

$$\begin{aligned}
b_1 &= \frac{c_2^3 - 3c_2^2/2 + 2c_2/3 - 1/9}{c_2^2(c_2-1)}, \\
b_2 &= \frac{c_2/2 - 1/6}{c_2^2}, \\
b_3 &= \frac{1}{18c_2^2(1-c_2)}.
\end{aligned}$$

d) We find an error estimation method by letting

$$\hat{b}_1 + \hat{b}_2 = 1, \quad \hat{b}_2 c_2 = \frac{1}{2}.$$

Recall that $c_2 \notin \{0,1\}$, so we can set

$$\hat{b}_1 = 1 - \frac{1}{2c_2}, \quad \hat{b}_2 = \frac{1}{2c_2}.$$

This is also a family parametrised by c_2 from the previous task.

Task 5

a) The eigenvalues are $\lambda_{1,2} = -10 \pm 20\mathrm{i}$.

b) The stability function is given by

$$R(z) = 1 + z + \frac{1}{2}z^2.$$

We must have

$$|R(h\lambda)| \leq 1$$

for the numerical solution to be stable for all eigenvalues of M. In our case, this means that

$$\begin{aligned}
\big|R\big((-10+20\mathrm{i})h\big)\big|^2 &= R\big((-10+20\mathrm{i})h\big) \cdot R\big((-10-20\mathrm{i})h\big) \\
&= 1 - 20h + 200h^2 - 5000h^3 + 62500h^4 \leq 1,
\end{aligned}$$

which is satisfied when $0 \leq h \leq 0.08603$.

c) See `ov07_5c.m`. Use for instance the step lengths 0.085, 0.086 and 0.087.

Task 6

You may do this task yourself. The results may surprise you!