TMA4215 Numerical Mathematics

Autumn 2010

Exercise 7

Task 1

Given an ordinary differential equation

$$y' = f(t, y), \qquad y(t_0) = y_0, \qquad t_0 \le t \le t_{\text{end}}. \tag{1}$$

You can assume that f satisfies the Lipschitz condition

$$\|f(t, y) - f(t, \tilde{y})\| \le L\|y - \tilde{y}\|.$$

A *one-step method* for solving this differential equation can be described by

$$y_{n+1} = y_n + h\Phi(t_n, y_n; h), \qquad n = 0, 1, \dots, N-1, \quad h = \frac{t_{\text{end}} - t_0}{N} \tag{2}$$

Assume the following:

- The local truncation error given by

  $$d_{n+1} = y(t_{n+1}) - y(t_n) - h\Phi\big(t_n, y(t_n); h\big)$$

  satisfies

  $$\|d_{n+1}\| \le Dh^{p+1}$$

  where D is a positive constant.

- The function Φ is Lipschitz continuous, with Lipschitz constant M, i.e.

  $$\|\Phi(t_n, y; h) - \Phi(t_n, \tilde{y}; h)\| \le M\|y - \tilde{y}\|. \tag{3}$$

a) Show that in this case, the global error in t_{end} satisfies

$$\|e_N\| = \|y(t_{\text{end}}) - y_N\| \le Ch^p,$$

where C is a positive constant depending on M, D and the interval $t_{\text{end}} - t_0$.

b) Assume that a two-stage explicit Runge–Kutta method given by the Butcher tableau

$$\begin{array}{c|cc} 0 & & \\ c_2 & c_2 & \\ \hline & b_1 & b_2 \end{array}$$

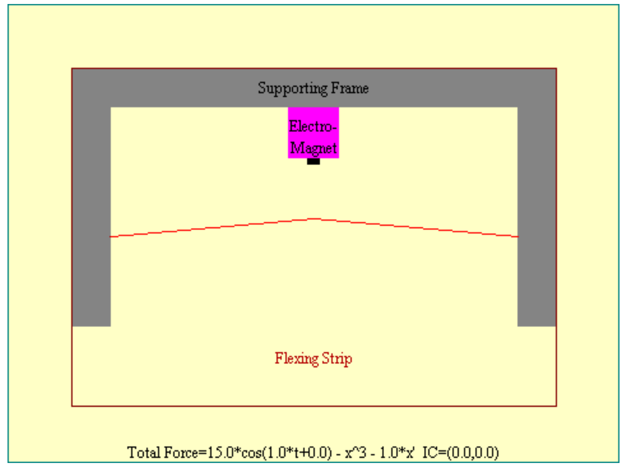
is used to solve (1). Show that the method can be written on the form (2). Now assume that $h \le h_{\max}$ and show that Φ satisfies the Lipschitz condition in y, with Lipschitz constant M that depends on the method coefficients c_2, b_1 and b_2, as well as L and $h_{\max}$.

Task 2

The Duffing oscillator is a much studied mathematical model. This can be described by the initial value problem

$$u'' + ku' - u(1 - u^2) = A\cos(\omega t). \quad (4)$$

In 1918, G. Duffing used this equation to describe a thin, flexible metal bar oscillating near an electromagnet. The constant k is the damping, while ω and A are the frequency and the amplitude of the driving force from the electromagnet respectively. See `http://www.mcasco.com/pattr1.html` for more details.



a) Start by transforming (4) to a system of two first-order differential equations.

b) Calculate by hand (you are allowed to use a calculator) a single step with the improved Euler method (also known as Heun's method), setting $k = 0.25$, $A = 0.4$, $\omega = 1.0$, $u(0) = 0$, $u'(0) = 0$, and using step length $h = 0.1$.

c) Implement the improved Euler method in MATLAB and use it to solve (4).

d) Create a plot of the first component u along the x-axis and the second component u' along the y-axis (this is called a *phase plot*). Start with the same parameters as in **b)**, but vary them and see what happens. You may use $h = 0.01$. Try integrating over quite long time intervals.

e) Try several different initial values and plot the resulting integral curves to get a picture of what the curves look like. You can use the same values as above for $k = 0.25$, $A = 0.4$, $\omega = 1.0$.

f) Finally, make an implementation where you replace improved Euler by RK4. Compare the results.

Task 3

Kutta's method from 1901 is the most famous of all explicit Runge–Kutta pairs, given by the following Butcher tableau:

$$
\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & \\
1 & 0 & 0 & 1 & \\
\hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{array}
$$

a) Verify that the method has order 4 by checking all 8 order conditions.

b) An alluring thought is to now find a new set of weights, say $\hat{b}_s$ such that the accompanying method is of order 3, for error estimates and step length control. Try to find such a set of $\hat{b}_s$.

Task 4

a) Show that an explicit Runge–Kutta method with s stages maximally can be of order s. (Hint: Use $y' = y$, $y(0) = y_0$ as test equation.)

b) Show that an explicit 3rd order Runge–Kutta method with 3 stages must satisfy

$$3a_{32}c_2^2 - 2a_{32}c_2 - c_2c_3 + c_3^2 = 0.$$

c) Characterise all 3rd order explicit Runge–Kutta methods with 3 stages that satisfy $a_{31} = 0$, i.e. $a_{32} = c_3$. How many free parameters are there?

d) Find all explicit methods of order 2 that have the same coefficients a_{ij} as the method above, and weights that simultaneously satisfy $\hat{b}_3 = 0$.

Task 5

a) Find the eigenvalues of the matrix

$$M = \begin{pmatrix} -10 & -10 \\ 40 & -10 \end{pmatrix}.$$

b) Assume that you are to solve the differential equation

$$y' = My, \qquad y(0) = y_0$$

using the improved Euler method. What is the largest step size $h_{\max}$ you can use?

c) Solve the equation

$$y' = My + g(t), \qquad 0 \leq t \leq 10$$

with

$$g(t) = \big(\sin(t), \cos(t)\big)^{\mathrm{T}}, \qquad y(0) = \left(\frac{5210}{249401}, \frac{20259}{249401}\right)^{\mathrm{T}}$$

by using `impEuler.m`. Choose step sizes a little smaller than and a little larger than $h_{\max}$. What do you observe?

Task 6

The linear test equation

$$y' = \lambda y, \qquad y(0) = y_0$$

has solution $y(h) = \mathrm{e}^z y_0$ where $z = \lambda h$. One step with a Runge–Kutta method gives $y_1 = R(z)y_0$. Thus, we can consider the stability function $R(z)$ as an approximation of e^z. Will $R(z)$ grow (absolutely) faster than e^z? We can find this out by studying when $|R(z)/\mathrm{e}^z| > 1$.

Rewrite the script `stab.m` so that it plots the region

$$\mathcal{A} = \{z \in \mathbb{C} \mid |R(z)/\mathrm{e}^z| > 1\}.$$

Calculate the stability function for some of the Runge–Kutta methods you know and find $\mathcal{A}$ for them. You may also draw the stability functions for the Gauss–Legendre methods (collocation methods of order $2s$). These are given by:

$$\begin{aligned} s &= 1, & R(z) &= \frac{1 + z/2}{1 - z/2}, \\ s &= 2, & R(z) &= \frac{1 + z/2 + z^2/12}{1 - z/2 + z^2/12}, \\ s &= 3, & R(z) &= \frac{1 + z/2 + z^2/10 + z^3/120}{1 - z/2 + z^2/10 - z^3/120}. \end{aligned}$$

The region $\mathcal{A}$ is called an *order star*.