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Exam in TMA4215

August 17th 2011, Time: 9:00-13:00

Hjelpemidler code C: Textbook Kincaid and Cheney, Numerical Analysis, third edition. TMA4215 lecture notes (39 pages).

Problem 1

a) Is the following function a natural cubic spline?

$$S(x) = \begin{cases} x^3 - 1 & x \in [-1, \frac{1}{2}] \\ 3x^3 - 1 & x \in [\frac{1}{2}, 1] \end{cases}$$

Justify your answer.

b) What values of (a, b, c, d) make the following a cubic spline?

$$f(x) = \begin{cases} x^3 & x \in [-1, 0] \\ a + bx + cx^2 + dx^3 & x \in [0, 1] \end{cases}$$

Problem 2

a) Find the first two polynomials orthogonal with respect to the inner product

$$\langle f, g \rangle_w = \int_0^1 w(x)\, f(x) g(x)\, dx,$$

where $w(x) = x$ in $[0, 1]$ is the weight function. Verify that the third orthogonal polynomial is

$$P_2(x) = \sqrt{3}\,(10x^2 - 12x + 3).$$

Note that we have chosen the normalization $\langle P_2, P_2 \rangle_w = 1$, use the same normalization for the other two polynomials P_0 and P_1.

b) Find a quadrature formula of the form

$$\int_0^1 x f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

with $n = 1$, that is exact for all polynomials of degree 3.

c) Use this formula to approximate the integral

$$\int_0^1 x \sin(x)\,dx.$$

Compute the error subtracting the resulting approximation from the exact integral.

Problem 3

a) We want to find the local error σ_{n+1} of the method

$$y_{n+1} = y_n + \frac{1}{2} h\big(f(y_n) + f(y_n + hf(y_n))\big),$$

for the numerical solution of the autonomous, scalar initial value problem $y'(t) = f(y(t))$, with $y(0) = y_0$, and where $h = t_{n+1} - t_n$.

We use the following definition of the local truncation error

$$\sigma_{n+1} = y(t_{n+1}) - z_{n+1},$$

with z_{n+1} defined by

$$z_{n+1} = y(t_n) + \frac{1}{2} h\big(f(y(t_n)) + f(y(t_n) + hf(y(t_n)))\big),$$

and it is sufficient to investigate the case $n = 0$.

Explain how we obtain the following expression for σ_1

$$\sigma_1 = h^3(C_1 f''(y_0)[f(y_0)]^2 + C_2[f'(y_0)]^2 f(y_0)) + \mathcal{O}(h^4)$$

find the constants C_1 and C_2 [1].

b) We will use the method for the numerical solution of the initial value problem

$$y'' = y'\,y, \quad y(0) = 1, \quad y'(0) = 0.5.$$

Reformulate the problem into a system of first order differential equations. Do one step with the proposed method to find the numerical approximations to $y(0.1)$ and $y'(0.1)$.

[1] If one considers $f : \mathbf{R} \to \mathbf{R}$, then $f'(y) = \frac{d}{dy} f(y)$ and $f''(y) = \frac{d^2}{dy^2} f(y)$. Otherwise for $f : \mathbf{R}^k \to \mathbf{R}^k$, $f'(y)$ is the Jacobian of f and f'' is the second differential.