TMA4215 Numerical Mathematics

Autumn 2011

Solution 2

Task 1

We consider the sequence specified in Definition 1.4 with $\varepsilon_k = |x_k - 0| = x_k$.

$$\begin{aligned} \frac{x_{k+1}}{x_k} &= \frac{2^{-(k+1)^\alpha}}{2^{-k^\alpha}} \\ &= 2^{k^\alpha-(k+1)^\alpha} \end{aligned}$$

We consider the exponent and rewrite it by the generalized binomial series

$$\begin{aligned} k^\alpha - (k+1)^\alpha &= k^\alpha - k^\alpha\left(1+\frac{1}{k}\right)^\alpha \\ &= k^\alpha - k^\alpha\left(1+\alpha k^{-1} + \binom{\alpha}{2}k^{-2} + \cdots\right) \\ &= -\alpha k^{\alpha-1} - \binom{\alpha}{2}k^{\alpha-2} - \cdots \end{aligned}$$

The series converges for $k > 1$, and we see that the dominating term when $k \to \infty$ is $-\alpha k^{\alpha-1}$. Therefore

$$\lim_{k\to\infty} k^\alpha - (k+1)^\alpha = \begin{cases} 0, & \alpha < 1, \\ -1, & \alpha = 1, \\ -\infty & \alpha > 1. \end{cases}$$

Since 2^x is continuous for all $x \in \mathbb{R}$ and $\lim_{x\to-\infty} 2^x = 0$,

$$\mu = \lim_{k\to\infty}\frac{x_{k+1}}{x_k} = \begin{cases} 1, & \alpha < 1, \\ \frac{1}{2}, & \alpha = 1, \\ 0, & \alpha > 1. \end{cases}$$

According to the definition, the (x_k) converges sublinearly when $\alpha < 1$, linearly when $\alpha = 1$ and superlinearly when $\alpha > 1$

Task 2

From the note on nonlinear equations, we know that it is sufficient to show the two conditions

$$G(D) \subseteq D \tag{1}$$

$$\max_i \sum_{j=1}^{3} \bar{g}_{ij} < 1, \qquad \text{where} \qquad \left|\frac{\partial g_i}{\partial x_j}(x)\right| \leq \bar{g}_{ij} \quad \text{for } x \in D. \tag{2}$$

It is relatively easy to see that

$$g_1(1,1,x_3) \approx 0.34 < g_1(x_1,x_2,x_3) \le 0.5 = g_1(0,x_2,x_3)$$
$$g_2(0,x_2,-1) \approx -0.048 < g_2(x_1,x_2,x_3) < 0.09 \approx g_2(1,x_2,1)$$
$$g_3(-1,1,x_3) \approx -0.61 < g_3(x_1,x_2,x_3) < -0.49 \approx g_2(1,1,x_3)$$

so (1) is satisfied. Likewise, we can show that

$$\left|\frac{\partial g_1}{\partial x_1}\right| < 0.281 \qquad \left|\frac{\partial g_1}{\partial x_2}\right| < 0.281 \qquad \left|\frac{\partial g_1}{\partial x_3}\right| = 0$$
$$\left|\frac{\partial g_2}{\partial x_1}\right| < 0.067 \qquad \left|\frac{\partial g_2}{\partial x_2}\right| = 0 \qquad \left|\frac{\partial g_2}{\partial x_3}\right| < 0.119$$
$$\left|\frac{\partial g_3}{\partial x_1}\right| < 0.136 \qquad \left|\frac{\partial g_3}{\partial x_2}\right| < 0.136 \qquad \left|\frac{\partial g_3}{\partial x_3}\right| = 0$$

for all $x \in D$. This means that

$$\max_i \sum_{j=1}^{3} \bar{g}_{ij} = \max\{0.562, 0.186, 0.272\} = 0.562 < 1$$

so condition (2) is also satisfied. Test this numerically yourself.

Task 3

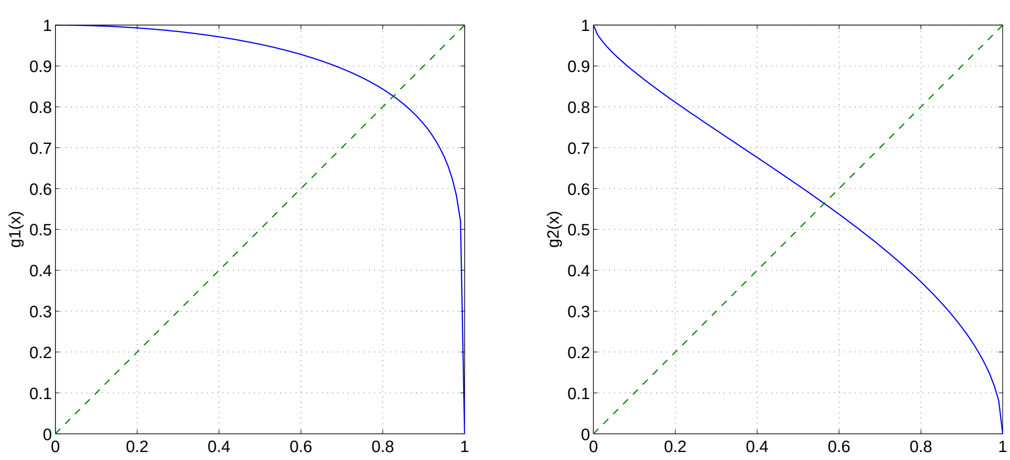
The fixed point iterations are given by

$$x_1^{(k+1)} = \sqrt[3]{x_2^{(k)}} \qquad x_1^{(k+2)} = \sqrt[6]{1-[x_1^{(k)}]^2}$$
$$x_2^{(k+1)} = \sqrt{1-[x_1^{(k)}]^2} \qquad x_2^{(k+2)} = \sqrt{1-[x_2^{(k)}]^{2/3}}$$

so we can view this as fixed point iterations on two scalar equations:

$$x = g_1(x) = \sqrt[6]{1-x^2}, \qquad x = g_2(x) = \sqrt{1-x^{2/3}}.$$

Start by locating the fixed points. This is easily done graphically:



This shows that g_1 has a fixed point near 0.8, and g_2 one near 0.5. For each of these, we must now find an interval $[a, b]$ so that *i)* $g_i([a, b]) \subseteq [a, b]$ and *ii)* $|g_i'(x)| < 1$ for $x \in [a, b]$.

Let us look at g_1 first. We see that

$$g_1'(x) = -\frac{x}{3(1-x^2)^{5/6}}, \qquad |g'(x)| < 1 \text{ for } 0 \leq x \leq 0.87.$$

But this interval does not satisfy *i)*. However, g_1 is monotonically decreasing. After a little trial and error, we find

$$g_1([0.76, 0.87]) \subseteq [0.76, 0.87].$$

Similarly, we can show that the two conditions are satisfied for g_2 on the interval $[0.22, 0.80]$. Thus, we have proven that the equation has a unique fixed point in the region

$$D = \{x \in \mathbb{R}^2 : 0.76 \leq x_1 \leq 0.87, 0.22 \leq x_2 \leq 0.80\}$$

and the iterations converge for all starting values in this region.

Task 4

Rewrite the iteration scheme on the form

$$Q\mathbf{x}^{(k+1)} = (Q - A)\mathbf{x}^{(k)} + b$$

with

$$Q = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -4 \end{bmatrix}, \qquad (Q - A) = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix}, \qquad \text{and} \qquad b = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

and $\mathbf{x}^{(k)} = [x_k, y_k, z_k]^T$. Find $T = Q^{-1}(Q - A)$ og A, and show $\|T\|_\infty = 0.75$. Thus the iteration scheme converges for all starting values. Further, $\lim_{k\to\infty} \mathbf{x}^{(k)} = \mathbf{x}$, where $\mathbf{x}$ is the solution of $A\mathbf{x} = b$. The exact solution in this case is $\boldsymbol{x} = [1/9, 1/9, -4/3]$. You can find this by iterating until convergence, or by solving the system using Gaussian elimination.

By the use of theorem 1.1 from the note on nonlinear equations, using $D = \mathbb{R}^3$ and $L = \|T\|_\infty$ we get

$$\|\mathbf{x}^{(k)} - \mathbf{x}\|_\infty \leq \frac{\|T\|_\infty}{1 - \|T\|_\infty} \|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty$$

or

$$\|\mathbf{x}^{(k)} - \mathbf{x}\|_\infty \leq \frac{\|T\|_\infty^k}{1 - \|T\|_\infty} \|\boldsymbol{x}^{(1)} - \mathbf{x}^{(0)}\|_\infty \leq 10^{-4}$$

Do one iteration to get $\mathbf{x}^{(1)}$, insert $\|T\|_\infty$, and see that $k = 37$ is sufficient. Such bounds are almost always very conservative, so in practice less iterations are needed.

Task 5

a)

$$x^{(1)} = \begin{bmatrix} 1.6 \\ 0.5 \\ 1.26 \end{bmatrix}, \qquad x^{(2)} = \begin{bmatrix} 1.08 \\ 1.06 \\ 1.06 \end{bmatrix}, \qquad x^{(3)} = \begin{bmatrix} 0.96 \\ 1.03333 \\ 0.98267 \end{bmatrix}$$

The iterations seem to converge, which is reasonable since the matrix is strictly diagonally dominant.

b)

$$x^{(1)} = \begin{bmatrix} 1.6 \\ -5.3 \\ -17.3 \end{bmatrix}, \qquad x^{(2)} = \begin{bmatrix} 9.20 \\ -115.1 \\ -339.1 \end{bmatrix}, \qquad x^{(3)} = \begin{bmatrix} 153.07 \\ -2155.7 \\ -6317.0 \end{bmatrix}$$

The iterations diverge. The spectral radius of the iteration matrix can be found to be $\rho(T) = 18.58$ using MATLAB, so divergence is reasonable.

Notice that the equations are the same, they are only permuted.

Task 6

See the suggested solution to the exam.