TMA4215 Numerical Mathematics

Autumn 2011

Exercise 2

Task 1

Consider the sequence

$$x_k = 2^{-k^\alpha}, \qquad k = 1, 2, 3, \ldots$$

where $\alpha > 0$. It is easily seen that

$$\lim_{k\to\infty} x_k = 0$$

Use Definition 1.4 in S&M and discuss the convergence of (x_k) for different α. When does the sequence converge linearly? Superlinearly? Sublinearly?

Extra: Use Definition 1.7 in S&M. Does the sequence converge with order $q > 1$ for any α?

Task 2

Given

$$G(x) = \begin{pmatrix} \frac{1}{3}\cos(x_1 x_2) + \frac{1}{6} \\ \frac{1}{9}\sqrt{x_1^2 + \sin x_3 + 1.06} - 0.1 \\ -\frac{1}{20}\mathrm{e}^{-x_1 x_2} - \frac{10\pi - 3}{60} \end{pmatrix}$$

Show that the fixed point iterations $x^{(k+1)} = G(x^{(k)})$ converge towards a unique fixed point for all starting values $x^{(0)}$ in $D = \{x \in \mathbb{R}^3 : -1 \le x_i \le 1,\ i = 1, 2, 3\}$.

Verify the result numerically.

Task 3

Consider the system of equations from the exercise 1,

$$
\begin{aligned}
x_1^2 + x_2^2 &= 1, \\
x_1^3 - x_2 &= 0.
\end{aligned}
$$

This has two solutions, one in the region $-1 \leq x_1, x_2 \leq 0$ and one in $0 \leq x_1, x_2 \leq 1$. It is possible to show numerically that the iteration scheme based on the formulation

$$
\begin{aligned}
x_1 &= \sqrt[3]{x_2}, \\
x_2 &= \sqrt{1 - x_1^2}
\end{aligned}
$$

converges with appropriate starting values.

Explain why. How would you select starting values?

Hint: It is simpler to analyse results if you consider two subsequent iterations as one.

Task 4

Given the iteration scheme:

$$
\begin{aligned}
4x_{k+1} &= -x_k - y_k + z_k + 2 \\
6y_{k+1} &= 2x_k + y_k - z_k - 1 \\
-4z_{k+1} &= -x_k + y_k - z_k + 4
\end{aligned}
$$

Prove that $\mathbf{x}^{(k)} = [x_k, y_k, z_k]^T$ converges to a limit $\mathbf{x}$ for all starting values $\mathbf{x}^{(0)}$ when $k \to \infty$. What is the limit $\mathbf{x}$?

How many iterations are needed to ensure $\|\mathbf{x}^{(k)} - \mathbf{x}\|_\infty \leq 10^{-4}$ if $\mathbf{x}^{(0)} = [0, 0, 0]^T$? Is this number realistic, or do you think that less iterations are needed in practice?

Task 5

Solve the two systems of equations by Gauss–Seidel iterations:

$$\begin{aligned} 3x + y + z &= 5 \\ x + 3y - z &= 3 \\ 3x + y - 5z &= -1 \end{aligned} \tag{1}$$

$$\begin{aligned} 3x + y + z &= 5 \\ 3x + y - 5z &= -1 \\ x + 3y - z &= 3. \end{aligned} \tag{2}$$

Use $[0.1, 0.1, 0.1]^T$ as the starting point. First do a few iterations by hand, and then you can use the MATLAB program `gs.m`.

Comment the results. Do they comply with theory?

Task 6

Exam problem: December 2008, problem 5.