TMA4215 Numerical Mathematics

Autumn 2011

Exercise 4

Task 1

a) Use divided differences and Newton's interpolation formula to find the interpolating polynomial of lowest possible degree for the points in the table:

x_i	0	2	3
y_i	1	2	4

b) Add the point $(1, 0)$ to the table in **a)**. What is the new interpolating polynomial?

Task 2

In this task, you are to approximate $\sin(x)$ on the interval $[0, \pi]$ using polynomial interpolation.

a) Choose 4 equidistant nodes on the interval. Find the polynomial and an upper bound for the error $|\sin(x) - p_3(x)|$.

b) Repeat **a)**, but use Chebyshev nodes instead (see hint). (You do not have to calculate the polynomial, but show the interpolation points.)

c) In the two cases (equidistant and Chebyshev nodes), find an expression for an upper error bound, expressed by n (the polynomial degree). Plot the error bound as a function of n and compare the two cases.

Hint: The Chebyshev nodes are defined on the interval $[-1, 1]$. To move them over to another interval $[a, b]$, the change of variables

$$x = \frac{a+b}{2} + \frac{b-a}{2}t, \qquad t \in [-1, 1]$$

is used. The error formula is adjusted in the same way.

Task 3

Write two MATLAB functions, one that calculates the table of divided differences based on a given dataset, and one that calculates the value of the interpolating polynomial in given points, based on this table. E.g.

```
function tab = divdiff(x, y)
```

and

```
function y = pval(tab, t)
```

where `t` may be a vector.

Iterated multiplication may be useful in this task. Let

$$p(x) = f_0 + f_1(x - x_0) + f_2(x - x_0)(x - x_1) + \cdots + f_n(x - x_0)(x - x_1) \cdots (x - x_{n-1}),$$

and define the polynomials $b_n, b_{n-1}, \ldots, b_0$ by

$$\begin{aligned} b_n(x) &= f_n \\ b_{n-1}(x) &= f_{n-1} + (x - x_{n-1})b_n(x) \\ &\vdots \\ b_0(x) &= f_0 + (x - x_0)b_1(x) \end{aligned}$$

Then $b_0(x) = p(x)$. This equality is easily seen by substituting for $b_1(x), b_2(x), \ldots, b_n(x)$ in the expression for b_0 and expand the expression. Test the functions on the dataset in Task 1.

Task 4

Given Runge's function $f(x) = 1/(1 + 25x^2)$, $x \in [-1, 1]$. Find and plot the polynomial interpolating f in equidistant nodes. Use $n = 6, 11$ and 21. Use the `polyfit` function in MATLAB. This function generates a polynomial of degree n interpolating given data. If we use $n + 1$ data points, and the degree of the polynomial is n, the obtained polynomial is the interpolation polynomial. Use the function `polyval` to evaluate the obtained (interpolation) polynomial on a desired range of values, and then plot the results.

Repeat the experiment with Chebyshev nodes and comment the result.

Task 5

Let the distance h between nodes be such that $x_i = a + ih$, $i = 0, 1, 2, \ldots$ Let $f_i = f(x_i)$, $i = 0, 1, 2, \ldots$

On such a sequence $\{f_i\}_{i=0}^{\infty}$, we can define a *forward difference* recursively by

$$\Delta^0 f_0 = f_0, \quad \Delta f_0 = f_1 - f_0, \qquad \Delta^k f_0 = \Delta(\Delta^{k-1} f_0) = \Delta^{k-1} f_1 - \Delta^{k-1} f_0, \qquad k = 1, 2, \ldots$$

We get

$$\Delta^2 f_0 = f_2 - 2f_1 + f_0, \qquad \Delta^3 f_0 = f_3 - 3f_2 + 3f_1 - f_0, \qquad \text{and so on.}$$

Let $x = x_0 + sh$, where $s \in \mathbb{R}$. The task is about showing that the polynomial interpolating f in the nodes x_i, $i = 0, 1, \ldots, n$ can be written

$$p_n(x) = p_n(x_0 + sh) = f_0 + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f_0 \tag{1}$$

where

$$\binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k!}.$$

a) Show by induction:

$$f[x_0, x_1, \ldots, x_k] = \frac{1}{k!h^k} \Delta^k f_0.$$

b) Show that

$$\prod_{i=0}^{k-1} (x - x_i) = k!h^k \binom{s}{k}, \qquad k \geq 1.$$

c) Use the results from **a)** and **b)** to prove Newton's forward difference formula (1).

d) Apply the formula to the dataset in Task **1b)**.

Comment: Equivalently, it is possible to show *Newton's backward difference formula.* Backward differences on the sequence $\{f_n\}_{n=0}^{\infty}$ are defined by

$$\nabla^0 f_n = f_n, \qquad \nabla f_n = f_n - f_{n-1}, \qquad \nabla^k f_n = \nabla^{k-1} f_n - \nabla^{k-1} f_{n-1}, \qquad k = 1, 2, \ldots$$

Newton's backward difference formula is given by

$$p_n(x) = p_n(x_n + sh) = f_n + \sum_{k=1}^{n} (-1)^k \binom{-s}{k} \nabla^n f_n.$$

Some relevant suggested exam problems:

Those of you that want some more problems/examples can take a look at the following exam problems:

December 2007, problem 1.
December 2006, problem 1.
August 2005, problem 2.