TMA4215 Numerical Mathematics

Autumn 2011

Solution 5

Task 1

We study Hermite interpolation, which is characterised by a polynomial $p(x)$ defined on $n+1$ distinct nodes $x_0, x_1, \ldots, x_n$ satisfying the conditions

$$p(x_i) = y_i, \quad p'(x_i) = v_i, \quad i = 0, 1, \ldots, n \tag{1}$$

where $\{y_i\}_{i=0}^n$ and $\{v_i\}_{i=0}^n$ are arbitrary, specified values.

a) It is reasonable to assume $p \in \mathbb{P}_{2n+1}$ since (1) specifies $2n+2$ conditions (2 conditions for each of the $n+1$ points). A polynomial of degree $2n+1$ can generally be represented by

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_{2n} x^{2n} + a_{2n+1} x^{2n+1}$$

and thus has $2n+2$ parameters $a_0, a_1, \ldots, a_{2n+1}$. Hence, we can use the conditions (1) to uniquely determine these parameters.

b) We assume that the functions $A_i(x)$ and $B_i(x)$ (which are not specified for now), all defined for $i = 0, 1, \ldots, n$, satisfy

$$\begin{aligned} A_i(x_j) &= \delta_{ij}, \quad & B_i(x_j) &= 0, \\ A_i'(x_j) &= 0, \quad & B_i'(x_j) &= \delta_{ij} \end{aligned} \tag{2}$$

for all $i, j = 0, 1, \ldots, n$. We define the function $g(x)$ as

$$g(x) = \sum_{i=0}^{n} y_i A_i(x) + \sum_{i=0}^{n} v_i B_i(x). \tag{3}$$

Note: We have not yet said *anything* about which type of function $A_i(x)$ and $B_i(x)$ are, just that they shall satisfy the conditions (2).

Given the function $g(x)$ in (3) we find for arbitrary $j = 0, 1, \ldots, n$

$$\begin{aligned} g(x_j) &= \sum_{i=0}^{n} y_i A_i(x_j) + \sum_{i=0}^{n} v_i B_i(x_j) = \sum_{i=0}^{n} y_i \delta_{ij} + \sum_{i=0}^{n} v_i \cdot 0 = y_j, \\ g'(x_j) &= \sum_{i=0}^{n} y_i A_i'(x_j) + \sum_{i=0}^{n} v_i B_i'(x_j) = \sum_{i=0}^{n} y_i \cdot 0 + \sum_{i=0}^{n} v_i \delta_{ij} = v_j. \end{aligned}$$

Thus, we have shown that a function $g(x)$ as defined in (3) satisfies (1) as long as the *basis functions* $A_i(x)$ and $B_i(x)$ satisfy (2).

c) We will now look at possible representations of the basis functions $A_i(x)$ and $B_i(x)$. Specifically, we look at basis functions $A_i \in \mathbb{P}_{2n+1}$ and $B_i \in \mathbb{P}_{2n+1}$. We use the cardinal functions

$$L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

which satisfy $L_i(x_j) = \delta_{ij}$ and study the functions

$$A_i(x) = \big(1 - 2(x - x_i)L_i'(x_i)\big)L_i^2(x), \quad B_i(x) = (x - x_i)L_i^2(x). \tag{4}$$

It is obvious that $A_i \in \mathbb{P}_{2n+1}$ and $B_i \in \mathbb{P}_{2n+1}$. Next, we find that

$$\begin{aligned} A_i'(x) &= -2L_i'(x_i)L_i^2(x) + 2\big(1 - 2(x - x_i)L_i'(x_i)\big)L_i(x)L_i'(x) \\ &= 2L_i(x)L_i'(x)\big(1 - L_i(x) - 2(x - x_i)L_i'(x_i)\big) \\ B_i'(x) &= L_i^2(x) + 2(x - x_i)L_i(x)L_i'(x) = L_i(x)\big(L_i(x) + 2(x - x_i)L_i'(x)\big). \end{aligned}$$

We see that the polynomials $A_i(x)$ and $B_i(x)$ satisfy (2) for all $i, j = 0, 1, \ldots, n$. Thus, the polynomials (4) are polynomials of degree $2n + 1$ which satisfy the necessary basis conditions and can be used as basis functions for construction of *polynomials* of degree $2n+1$ satisfying conditions (1).

d) We shall find a third degree polynomial $p(x)$ satisfying

$$p(1) = 1, \quad p'(1) = 3, \quad p(2) = 14, \quad p'(2) = 24.$$

In this case $2n + 1 = 3 \Rightarrow n = 1$ and we find

$$\begin{aligned} L_0(x) &= \frac{x-2}{1-2} = 2 - x, & L_1(x) &= \frac{x-1}{2-1} = x - 1 \\ L_0'(x) &= -1, & L_1'(x) &= 1 \\ L_0^2(x) &= x^2 - 4x + 4, & L_1^2(x) &= x^2 - 2x + 1 \end{aligned}$$

From the representation (4) we find

$$\begin{aligned} A_0(x) &= \big(1 - 2(x - 1)\cdot(-1)\big)(x^2 - 4x + 4) = (2x - 1)(x^2 - 4x + 4) \\ A_1(x) &= \big(1 - 2(x - 2)\cdot 1\big)(x^2 - 2x + 1) = (5 - 2x)(x^2 - 2x + 1) \\ B_0(x) &= (x - 1)(x^2 - 4x + 4) \\ B_1(x) &= (x - 2)(x^2 - 2x + 1). \end{aligned}$$

Thus, the third degree polynomial satisfying the conditions above is given by

$$\begin{aligned} p(x) &= (2x - 1)(x^2 - 4x + 4) + 3\cdot(x - 1)(x^2 - 4x + 4) \\ &\quad + 14\cdot(5 - 2x)(x^2 - 2x + 1) + 24\cdot(x - 2)(x^2 - 2x + 1) \\ &= x^3 + 6x^2 - 12x + 6. \end{aligned}$$

The final result follows easily from a little calculation.

Task 2

Let $s(x)$ be a spline function of the given type. Then $s(x)$ is a continuous function which on the interval $I_j := [x_j, x_{j+1}]$ coincides with some linear function $s_j(x) = \gamma_j(x - x_j) + \delta_j$.

We try and choose $\beta, \alpha_0, \alpha_1, \ldots$ such that there is equality in each interval I_j.

We see that $(x - x_j)^+$ is zero for $x \leq x_j$ and coincides with the line with slope 1 through x_j for $x > x_j$. On the interval I_j,

$$(x - x_i)^+ = \begin{cases} x - x_i, & i \leq j \\ 0, & i > j \end{cases},$$

so the right hand side of the equality is equal to $\beta + \sum_{i=0}^{j} \alpha_i(x - x_i)$ on I_j.

On I_0 we see that for the equality to hold, we need

$$s_0(x) = \gamma_0(x - x_0) + \delta_0 = \beta + \alpha_0(x - x_0),$$

so we set $\beta = \delta_0$, $\alpha_0 = \gamma_0$.

We proceed by induction and assume that equality holds on I_j, i.e.

$$s_j(x) = \gamma_j(x - x_j) + \delta_j = \beta + \sum_{i=0}^{j} \alpha_i(x - x_j).$$

For equality to hold on I_{j+1} we need

$$s_{j+1}(x) = \gamma_{j+1}(x - x_{j+1}) + \delta_{j+1} = s_j(x) + \alpha_{j+1}(x - x_{j+1})$$

We know that $s_j(x_{j+1}) = s_{j+1}(x_{j+1})$ since $s(x)$ is continuous, so the linear functions $\gamma_{j+1}(x - x_{j+1}) + \delta_{j+1}$ and $s_j(x) + \alpha_{j+1}(x - x_{j+1}) = \delta_j + \gamma_j(x - x_j) + \alpha_{j+1}(x - x_{j+1})$ are equal if they have the same slope. We set $\alpha_{j+1} = \gamma_{j+1} - \gamma_j$ to achieve this.

Task 3

Consider i so that $|s_i''| = \max_l |s_l''|$, and consider equation i in the system of linear equations used for calculating the natural cubic splines:

$$hs_{i-1}'' + 4hs_i'' + hs_{i+1}'' = 6\left(\frac{f_{i+1} - f_i}{h} - \frac{f_i - f_{i-1}}{h}\right).$$

By using that $|s_i''| = \max_l |s_l''|$ and Taylor's theorem, we have that

$$4h|s_i''| \leq h|s_{i-1}''| + h|s_{i+1}''| + 6|f'(\eta_i) - f'(\eta_{i-1})| \leq 2h|s_i''| + 6|f'(\eta_i) - f'(\eta_{i-1})|$$

and thus

$$h|s_i''| \leq 3|f'(\eta_i) - f'(\eta_{i-1})|,$$

with $\eta_i \in (x_i, x_{i+1})$, $\eta_{i-1} \in (x_{i-1}, x_i)$. Additionally we have

$$|f'(\eta_i) - f'(\eta_{i-1})| = \frac{|f'(\eta_i) - f'(\eta_{i-1})|}{|\eta_i - \eta_{i-1}|}|\eta_i - \eta_{i-1}| \leq |2hf''(\xi)| \leq 2h\|f''\|_\infty,$$

so that

$$|s_i''| \leq 6\|f''\|_\infty.$$

Application of the result in task 3

Under the assumptions of task 3, we now want to show the error bound

$$|f(x) - s(x)| \leq \frac{7}{8} h^2 \|f''\|_\infty. \tag{5}$$

Consider the interval $[x_{i-1}, x_i]$, and let $\bar{x} \in (x_{i-1}, x_i)$. Consider the function

$$g(x) := f(x) - s(x) - \frac{(x - x_i)(x - x_{i-1})}{(\bar{x} - x_i)(\bar{x} - x_{i-1})} \big(f(\bar{x}) - s(\bar{x})\big).$$

We have $g(x_i) = 0$, $g(x_{i-1}) = 0$ and $g(\bar{x}) = 0$, so $g''(x)$ has at least one zero $\bar{\xi}_i \in (x_{i-1}, x_i)$ so that we get

$$0 = f''(\bar{\xi}_i) - s''(\bar{\xi}_i) - \frac{2}{(\bar{x} - x_i)(\bar{x} - x_{i-1})} \big(f(\bar{x}) - s(\bar{x})\big),$$

and

$$f(\bar{x}) - s(\bar{x}) = \frac{(\bar{x} - x_i)(\bar{x} - x_{i-1})}{2} \big(f''(\bar{\xi}_i) - s''(\bar{\xi}_i)\big).$$

Since $s''(x)$ is a linear polynomial on $[x_{i-1}, x_i]$, (a straight line between (x_{i-1}, s''_{i-1}) and (x_i, s''_i)), we have

$$|s''(\bar{\xi}_i)| \leq \max\{|s''_{i-1}|, |s''_i|\}, \quad \forall i.$$

Also, one can easily show (see exercise 3, task 3) that

$$|(\bar{x} - x_i)(\bar{x} - x_{i-1})| \leq \frac{h^2}{4},$$

which leads to

$$|f(x) - s(x)| \leq \frac{1}{8} h^2 (\|f''\|_\infty + \max_i |s''_i|).$$

Together with the result of task 3, this gives (5).