TMA4215 Numerical Mathematics

Autumn 2011

Solution 6

Task 1

Let $R(n-1,0) = T(2h)$ and $R(n,0) = T(h)$, with $m = 2^n$, $h = (b-a)/m$, and

$$T(h) = h\left(\frac{1}{2}f(x_0) + \sum_{i=1}^{m-1} f(x_i) + \frac{1}{2}f(x_m)\right), \qquad x_i = a + ih, i = 0, \ldots, m.$$

The recursive trapezoid rule gives

$$T(h) = \frac{T(2h)}{2} + h\sum_{i=1}^{m/2} f(x_{2i-1}).$$

The Romberg algorithm gives

$$\begin{aligned} R(n,1) = \frac{4}{3}R(n,0) - \frac{1}{3}R(n-1,0) &= \frac{4}{3}\left(\frac{T(2h)}{2} + h\sum_{i=1}^{m/2} f(x_{2i-1})\right) - \frac{T(2h)}{3} \\ &= \frac{h}{3}\left(f(x_0) + 2\sum_{i=1}^{m/2-1} f(x_{2i}) + 4\sum_{i=1}^{m/2} f(x_{2i-1}) + f(x_m)\right). \end{aligned}$$

Task 2

There will be given no solution to this task. Also see the exam 2008, problem 2.

Task 3

Let $T_1 = T(a,b) = (b-a)\big(f(a)+f(b)\big)/2$, and $T_2 = T(a,c) + T(c,b)$, where $c = (a+b)/2$. We know that

$$\begin{aligned} \int_a^b f(x)\,\mathrm{d}x &= T_1 - \frac{(b-a)^3}{12}f''(\xi_1), \\ \int_a^b f(x)\,\mathrm{d}x &= T_2 - \frac{(b-a)^3}{12\cdot 2^3}\left(f''(\eta_1) + f''(\eta_2)\right), \end{aligned}$$

where $\xi_1 \in (a,b)$, $\eta_1 \in (a,c)$ and $\eta_2 \in (c,b)$. If we assume that $f''(x)$ changes little over the interval (a,b), we can use the same reasoning as for the adaptive Simpson's rule. An appropriate error estimate for T_2 is

$$\int_a^b f(x)\,\mathrm{d}x - T_2 \approx \mathcal{E}(a,b) = \frac{1}{3}(T_2 - T_1).$$

The adaptive trapezoid algorithm then becomes:

function ADAPTIVE-TRAPEZOID(f, a, b, tol)
    $T_1 \leftarrow T(a, b)$ $\triangleright\ T(a,b) = (b-a)\big(f(a)+f(b)\big)/2$
    $c \leftarrow (a+b)/2$
    $T_2 \leftarrow T(a,c) + T(c,b)$
    $\mathcal{E} \leftarrow (T_2 - T_1)/3$
    if $|\mathcal{E}| \le tol$ **then**
        return T_2
    else
        $T_l \leftarrow$ ADAPTIVE-TRAPEZOID($f, a, c, tol/2$)
        $T_r \leftarrow$ ADAPTIVE-TRAPEZOID($f, c, b, tol/2$)
        return $T_l + T_r$
    end if
end function

Applied to the integral in the text, the algorithm gives:

<table>
<tr><td colspan="4">$tol = 2 \cdot 10^{-3},\ a = 0.0,\ b = 0.8$
$T_1 = 0.23888,\ T_2 = 0.18317,\ \mathcal{E} = -1.86 \cdot 10^{-2}$</td></tr>
<tr><td colspan="2">$tol = 1 \cdot 10^{-3}\ a = 0.0,\ b = 0.4$
$T_1 = 0.031864,\ T_2 = 0.0239330$
$\mathcal{E} = -2.6 \cdot 10^{-3}$</td><td colspan="2">$tol = 1 \cdot 10^{-3}\ a = 0.4,\ b = 0.8$
$T_1 = 0.15130,\ T_2 = 0.14611$
$\mathcal{E} = -1.7 \cdot 10^{-3}$</td></tr>
<tr><td>$tol = 5 \cdot 10^{-4}$
$a = 0.0,\ b = 0.2$
$T_1 = 0.004000$
$T_2 = 0.003000$
$\mathcal{E} = -3.3 \cdot 10^{-4}$</td><td>$tol = 5 \cdot 10^{-4}$
$a = 0.2,\ b = 0.4$
$T_1 = 0.019931$
$T_2 = 0.018953$
$\mathcal{E} = -3.3 \cdot 10^{-4}$</td><td>$tol = 5 \cdot 10^{-4}$
$a = 0.4,\ b = 0.6$
$T_1 = 0.051159$
$T_2 = 0.050320$
$\mathcal{E} = -2.8 \cdot 10^{-4}$</td><td>$tol = 5 \cdot 10^{-4}$
$a = 0.6,\ b = 0.8$
$T_1 = 0.094947$
$T_2 = 0.094536$
$\mathcal{E} = -1.4 \cdot 10^{-4}$</td></tr>
</table>

So

$$T = 0.003 + 0.018953 + 0.050320 + 0.094536 = 0.1668.$$

Task 4

a)

$$\int_{-1}^{1} \frac{\mathrm{e}^x}{\sqrt{1-x^2}}\,\mathrm{d}x \approx \frac{\mathrm{e}^{-1/\sqrt{3}}}{\sqrt{1-1/3}} + \frac{\mathrm{e}^{1/\sqrt{3}}}{\sqrt{1-1/3}} = \sqrt{6}\cosh(1/\sqrt{3}) \approx 2.8692$$

b) Use the inner product $\langle f, g\rangle = \int_{-1}^{1}\big(f(x)g(x)/\sqrt{1-x^2}\big)\,\mathrm{d}x$, with Chebyshev polynomials as orthogonal polynomials. We choose $n = 1$, $T_2(x) = 2x^2 - 1$, i.e. $x_0 = -1/\sqrt{2}$, $x_1 = 1/\sqrt{2}$. The weights become

$$A_0 = \int_{-1}^{1} \frac{\big(x - 1/\sqrt{2}\big)}{\big(-1/\sqrt{2} - 1/\sqrt{2}\big)} \cdot \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{\pi}{2},$$

$$A_1 = \int_{-1}^{1} \frac{\big(x + 1/\sqrt{2}\big)}{\big(1/\sqrt{2} + 1/\sqrt{2}\big)} \cdot \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{\pi}{2}.$$

The approximation becomes $\frac{\pi}{2}(\mathrm{e}^{-1/\sqrt{2}} + \mathrm{e}^{1/\sqrt{2}}) \approx 3.9603$ which is significantly better than the answer in **a)**.

c) The error is given by

$$E = K \cdot f^{(4)}(\nu), \qquad K = \frac{1}{4!}\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\left(x^2 - \frac{1}{2}\right)^2 \mathrm{d}x = \frac{\pi}{192}.$$

Since $|\mathrm{e}^x| < \mathrm{e}$ on $(-1,1)$, we must have $|E| < \mathrm{e}\pi/192 \approx 0.044$. The measured error is 0.017.

Task 5

The recursion formula from the note gives

$$\begin{aligned}
&\phi_0(x) = 1, && \langle \phi_0, \phi_0 \rangle = \int_0^\infty \mathrm{e}^{-x}\,\mathrm{d}x = 1 && \\
& && \langle x\phi_0, \phi_0 \rangle = \int_0^\infty \mathrm{e}^{-x} x\,\mathrm{d}x = 1 && \Rightarrow B_1 = 1 \\
&\phi_1(x) = x - 1, && \langle \phi_1, \phi_1 \rangle = \int_0^\infty \mathrm{e}^{-x}(x-1)^2\,\mathrm{d}x = 1 && \\
& && \langle x\phi_1, \phi_1 \rangle = 3 && \Rightarrow B_2 = 3, C_2 = 1 \\
&\phi_2(x) = (x-3)\phi_1(x) - \phi_0(x) = x^2 - 4x + 3 - 1 = x^2 - 4x + 2 && &&
\end{aligned}$$

Task 6

We must show that

$$\int_{-1}^{1} \left((x^2-1)^k\right)^{(k)} \left((x^2-1)^j\right)^{(j)} \mathrm{d}x = 0 \quad \text{for all } j < k.$$

Partial integration, $\int_a^b u\,\mathrm{d}v = uv|_a^b - \int_a^b v\,\mathrm{d}u$ with

$$u = \left((x^2-1)^j\right)^{(j)}, \qquad \mathrm{d}v = \left((x^2-1)^k\right)^{(k)} \mathrm{d}x$$

applied to the integral above gives

$$\left((x^2-1)^k\right)^{(k-1)} \left((x^2-1)^j\right)^{(j)} \Big|_{-1}^{1} - \int_{-1}^{1} \left((x^2-1)^k\right)^{(k-1)} \left((x^2-1)^j\right)^{(j+1)} \mathrm{d}x.$$

The first term is zero (see hint). Again, we apply partial integration to the integral. After having done this $j+1$ times, we end up with

$$(-1)^{j+1} \int_{-1}^{1} \left((x^2-1)^k\right)^{(k-j-1)} \left((x^2-1)^j\right)^{(2j+1)} \mathrm{d}x = 0$$

since $\left((x^2-1)^j\right)^{(2j+1)} = 0$.