TMA4215 Numerical Mathematics

Autumn 2012

Exercise 1

You should have read 1.4, 1.5 and 4.3 in S&M before solving these exercises.

Task 1

a) Apply Newton's method to the equation $f(x) = 0$ where

i) $f(x) = \cos x - 1/2$, with $x_0 = 0.5$.
ii) $f(x) = \mathrm{e}^x - x - 1$, with $x_0 = 0.5$.
iii) $f(x) = x(1 - \cos x)$, with $x_0 = 0.5$.

Use MATLAB The iterations will converge to a zero $x^\star$ in all three cases. Measure the order of convergence using e.g. MATLAB in the three cases. Is the result in accordance with theory? If no, can you explain why? **Note:** For some of the equations you may encounter problems with machine precision. Relative errors of approximately the same magnitude as machine epsilon ($\approx 2.22 \cdot 10^{-16}$ are usually not due to the numerical method.

We say that a zero $x^\star$ of $f(x)$ has multiplicity m if there exists a function $q(x)$ such that

$$f(x) = (x - x^\star)^m q(x), \qquad q(x^\star) \neq 0,$$

which is the case if and only if

$$f(x^\star) = f'(x^\star) = \cdots = f^{(m-1)}(x^\star) = 0, \qquad f^{(m)}(x^\star) \neq 0.$$

b) What is the multiplicity of the solutions of the three equations in **a)**?

c) Assume that $x^\star$ is a zero with multiplicity m of the function $f(x)$. Show that the function

$$\mu(x) = f(x)/f'(x)$$

has a simple zero in $x^\star$, independent of m. Use this to find an iteration scheme that converges quadratically to $x^\star$.

d) Test the new scheme on the functions *ii)* and *iii)* in **a)**.

e) Repeat the task in **a)**using the secant method instead of Newton's method.

Task 2

Consider the system of equations

$$\begin{aligned} x_1^2 + x_2^2 &= 1, \\ x_1^3 - x_2 &= 0. \end{aligned}$$

This has two solutions, one in the region $-1 \le x_1, x_2 \le 0$ and one in $0 \le x_1, x_2 \le 1$.

a) Choose appropriate initial values and perform two iterations by hand using Newton's method.

b) Verify that you get correct answers using MATLAB.

c) Explain what happens when you choose initial value lying on the x_2-axis.

Task 3

Consider the linear system $Ax = b$, with A an $N \times N$ matrix, $x, b \in \mathbf{R}^N$ and $\det(A) \neq 0$. Let $x(\varepsilon)$ be the solution of the perturbed linear system

$$(A + \varepsilon\, F)\, x(\varepsilon) \;=\; (b + \varepsilon\, v),$$

where $\varepsilon > 0$ is small, F is an $N \times N$ matrix and $v \in \mathbf{R}^N$.

a) Prove that if ε is small enough and $\det(A) \neq 0$ then $\det(A + \varepsilon F) \neq 0$, thus the solution $x(\varepsilon)$ of the perturbed linear system exists and is unique.
Hint. Verify that $\det(A + \varepsilon F)$ is a polynomial of degree N in ε, see what happens when $\varepsilon \to 0$.

Try first with the 2×2 case.

b) Prove that the components of $x(\varepsilon)$ are continuous functions of ε and differentiable in 0.
Hint. You might use the Cramer's rule for expressing the components of $x(\varepsilon)$ as quotients of determinants and then note that the involved determinants are polynomials of degree N in ε. Try first with the 2×2 case.