TMA4215 Numerical Mathematics

Autumn 2012

Solution 4

Task 1

a) The table of divided differences is ($f[x_i] = y_i$):

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$
0	1		
		1/2	
2	2		1/2
		2	
3	4		

and we end up with the polynomial

$$p_2(x) = 1 + \frac{1}{2}x + \frac{1}{2}x(x-2) = \frac{1}{2}x^2 - \frac{1}{2}x + 1.$$

b) We keep the table from **a)** and just add one more row:

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
0	1			
		1/2		
2	2		1/2	
		2		−1/2
3	4		0	
		2		
1	0			

The polynomial becomes

$$p_3(x) = 1 + \frac{1}{2}x + \frac{1}{2}x(x-2) - \frac{1}{2}x(x-2)(x-3) = -\frac{1}{2}x^3 + 3x^2 - \frac{7}{2}x + 1.$$

Task 2

a) Equidistant nodes: $x_i = ih$, $i = 0, \ldots, 3$ with $h = \pi/3$. We are to interpolate the points in the table

x_i	0	$\pi/3$	$2\pi/3$	π
$f(x_i)$	0	$\sqrt{3}/2$	$\sqrt{3}/2$	0

Result:

$$p_3(x) = \frac{9\sqrt{3}}{4\pi^2}(-x^2 + \pi x),$$

with error bound (see Exercise 3, Task 3)

$$|\sin(x) - p_3(x)| \leq \frac{1}{16}\left(\frac{\pi}{3}\right)^4 \approx 0.0752.$$

b) The Chebyshev nodes are: $x_i = \pi/2 + (\pi/2)\cos\big((2i+1)\pi/8\big)$ when $n = 3$. This gives us the table

x_i	3.022022903	2.171914057	0.9696785970	0.119569751
$f(x_i)$	0.1192850409	0.8247039815	0.8247039818	0.1192850413

The polynomial is

$$p_3(x) = -0.4043173324\,x^2 + 1.270200363\,x - 0.0268120051$$

with error bound

$$|\sin(x) - p_3(x)| \le \frac{1}{8 \cdot 4!}\left(\frac{\pi}{2}\right)^4 \approx 3.171 \cdot 10^{-2}.$$

c) We have $M_n = \max_{x \in [0,\pi]} |f^{(n+1)}(x)| = 1$.

Equidistant nodes:

$$|f(x) - p_n(x)| \le \frac{1}{4(n+1)}\left(\frac{\pi}{n}\right)^{n+1}$$

Chebyshev nodes: The change of variables leads to

$$\prod_{i=0}^{n}(x - x_i) = \left(\frac{b-a}{2}\right)^{n+1}\prod_{i=0}^{n}(t - t_i) \qquad \Rightarrow \qquad \prod_{i=0}^{n}|x - x_i| \le \left(\frac{b-a}{2}\right)^{n+1}\frac{1}{2^n}$$

so that the error bound becomes

$$|f(x) - p_n(x)| \le \left(\frac{\pi}{4}\right)^{n+1}\frac{2}{(n+1)!}.$$

In Figure 1 we see that Chebyshev nodes give lower error bound than equidistant nodes.

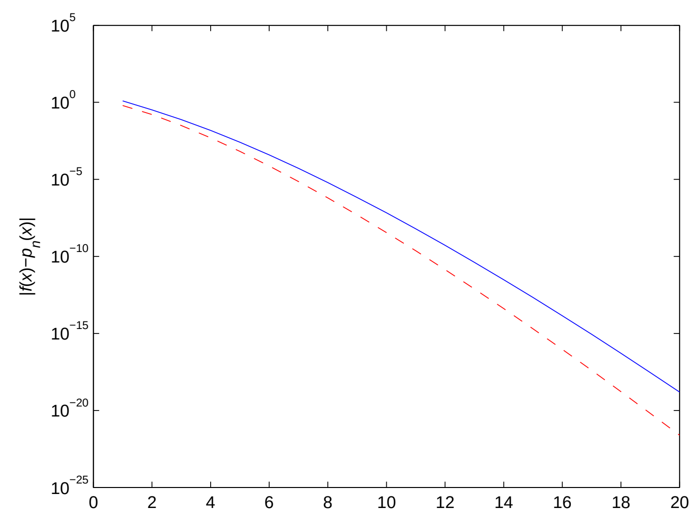

Figure 1: Solid: equidistant nodes, dashed: Chebyshev nodes.

Task 3

See the provided Matlab files.

Task 4

In Figure 2 we see that equidistant nodes give catastrophic results when approximating Runge's function. The larger we choose n, the larger displacement we get near -1 and 1. Chebyshev nodes, on the other hand, give us a good fit in the whole interval, and the larger we choose n, the better the fit.

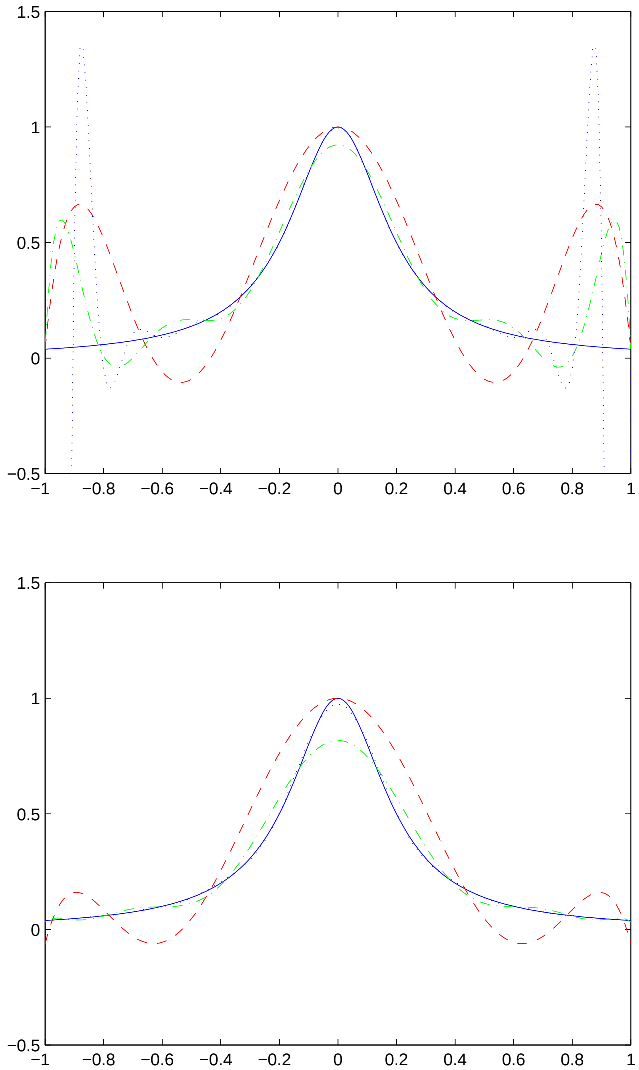

Figure 2: Top: equidistant nodes, bottom: Chebyshev nodes. Solid: Runge's function, dashed: $n = 6$, dashed/dotted: $n = 11$, dotted: $n = 21$.

Task 5

a) We have $f[x_0] = \Delta^0 f = f_0$ and $f[x_0, x_1] = (f_1 - f_0)/h$, so the assumption is correct for $k = 0, 1$. We now assume that the hypothesis is correct for some arbitrary k, and are to show that it then also is true for $k+1$. We have:

$$\begin{aligned} f[x_0, \dots, x_k, x_{k+1}] &= \frac{f[x_1, \dots, x_{k+1}] - f[x_0, \dots, x_k]}{x_{k+1} - x_0} \\ &= \frac{\frac{1}{k!h^k}(\Delta^k f_1 - \Delta^k f_0)}{(k+1)h} = \frac{1}{(k+1)!h^{k+1}} \Delta^{k+1} f_0. \end{aligned}$$

b) Since $x = x_0 + sh$, we have $x_i = x_0 + ih$ so that

$$\prod_{i=1}^{k-1} (x - x_i) = h^k \prod_{i=0}^{k-1} (s - i) = h^k k! \binom{s}{k}.$$

c) Insert the results from **a)** and **b)** into the known formula

$$p_n(x) = \sum_{k=0}^{n} f[x_0, \dots, x_k] \prod_{i=0}^{k-1} (x - x_i).$$

d) Sort the nodes in increasing order. Then we get the following table of forward differences:

$$\begin{array}{cccc} 1 & & & \\ & -1 & & \\ 0 & & 3 & \\ & 2 & & -3 \\ 2 & & 0 & \\ & 2 & & \\ 4 & & & \end{array}$$

and the polynomial becomes

$$p(x) = p(x_0 + sh) = 1 - \binom{s}{1} + 3\binom{s}{2} - 3\binom{s}{3} = 1 - s + 3\frac{s(s-1)}{2} - 3\frac{s(s-1)(s-2)}{3!},$$

which is the same polynomial as in Task **1b)**, since in this case $x_0 = 0$ and $h = 1$.