TMA4215 Numerical Mathematics

Autumn 2012

Exercise 5

Task 1

We will study *Hermite interpolation* in this task.

Given $n+1$ distinct nodes $x_0, x_1, \ldots, x_n$, we are to find a polynomial $p(x)$ of lowest possible degree satisfying

$$p(x_i) = y_i, \quad p'(x_i) = v_i, \quad i = 0, 1, \ldots, n \tag{1}$$

for arbitrary values y_i and v_i.

a) Why is it reasonable to assume that $p(x)$ will be of degree less than or equal to $2n+1$, i.e. $p \in \mathbb{P}_{2n+1}$?

b) Show that a function $g(x)$ given by

$$g(x) = \sum_{i=0}^{n} y_i A_i(x) + \sum_{i=0}^{n} v_i B_i(x)$$

satisfies the conditions (1) if the functions $A_i(x)$ and $B_i(x)$ satisfy

$$\begin{aligned} A_i(x_j) &= \delta_{ij}, \quad & B_i(x_j) &= 0, \\ A_i'(x_j) &= 0, \quad & B_i'(x_j) &= \delta_{ij} \end{aligned} \tag{2}$$

where $\delta_{ij} = 1$ when $j = i$ and else is 0.

c) Let $L_i(x)$ be the ordinary cardinal functions in Lagrange interpolation. Show that the following polynomials satisfy (2) for all $i = 0, 1, \ldots, n$:

$$A_i(x) = \big(1 - 2(x - x_i)L_i'(x_i)\big)L_i^2(x), \quad B_i(x) = (x - x_i)L_i^2(x).$$

d) Use this to find a third degree polynomial $p(x)$ satisfying

$$\begin{aligned} p(1) &= 1, \quad & p(2) &= 14 \\ p'(1) &= 3, \quad & p'(2) &= 24. \end{aligned}$$

Task 2

Show that a generic spline function $s(x)$ of degree 1 on $[a, b]$ and nodes $a = x_0 < x_1 < \cdots < x_n = b$, can be written

$$\beta + \sum_{i=1}^{n-1} \alpha_i (x - x_i)^+,$$

where

$$(x - x_i)^+ = \begin{cases} 0, & x \leq x_i, \\ x - x_i, & x_i < x. \end{cases}$$

Task 3

Consider a natural cubic spline function $s(x)$ on $[a, b]$ and equidistant nodes $a = x_0 < x_1 < \cdots < x_n = b$, $h = x_i - x_{i-1}$, $i = 1, \ldots, n$. Assume that s interpolates $f \in C^2[a, b]$ on $a = x_0 < x_1 < \cdots < x_n = b$. Show that

$$|s_i''| \leq 6\|f''\|_\infty,$$

where $s_i'' := s''(x_i)$ and $\|f''\|_\infty = \max_{x \in [a,b]} |f''(x)|$.

Hint: Use equation i in the system of equations that must be solved when constructing the natural cubic splines. Show and use the property

$$|f'(\eta_i) - f'(\eta_{i-1})| \leq 2h\|f''\|_\infty,$$

where $\eta_i \in (x_i, x_{i+1})$ and $\eta_{i-1} \in (x_{i-1}, x_i)$.

Task 4

Prove Theorem 11.3 p. 300 in S&M.

Hint: Read the proof of Theorem 11.3 p. 296.