

TMA4215 Numerical Mathematics

Autumn 2012

Solution 6

Task 1

Let $R(n-1, 0) = T(2h)$ and $R(n, 0) = T(h)$, with $m = 2^n$, $h = (b-a)/m$, and

$$T(h) = h \left(\frac{1}{2}f(x_0) + \sum_{i=1}^{m-1} f(x_i) + \frac{1}{2}f(x_m) \right), \quad x_i = a + ih, i = 0, \dots, m.$$

The recursive trapezoid rule gives

$$T(h) = \frac{T(2h)}{2} + h \sum_{i=1}^{m/2} f(x_{2i-1}).$$

The Romberg algorithm gives

$$\begin{aligned} R(n, 1) &= \frac{4}{3}R(n, 0) - \frac{1}{3}R(n-1, 0) = \frac{4}{3} \left(\frac{T(2h)}{2} + h \sum_{i=1}^{m/2} f(x_{2i-1}) \right) - \frac{T(2h)}{3} \\ &= \frac{h}{3} \left(f(x_0) + 2 \sum_{i=1}^{m/2-1} f(x_{2i}) + 4 \sum_{i=1}^{m/2} f(x_{2i-1}) + f(x_m) \right). \end{aligned}$$

Task 2

There will be given no solution to this task. Also see the exam 2008, problem 2.

Task 3

Let $T_1 = T(a, b) = (b-a)(f(a) + f(b))/2$, and $T_2 = T(a, c) + T(c, b)$, where $c = (a+b)/2$. We know that

$$\begin{aligned} \int_a^b f(x) dx &= T_1 - \frac{(b-a)^3}{12} f''(\xi_1), \\ \int_a^b f(x) dx &= T_2 - \frac{(b-a)^3}{12 \cdot 2^3} (f''(\eta_1) + f''(\eta_2)), \end{aligned}$$

where $\xi_1 \in (a, b)$, $\eta_1 \in (a, c)$ and $\eta_2 \in (c, b)$. If we assume that $f''(x)$ changes little over the interval (a, b) , we can use the same reasoning as for the adaptive Simpson's rule. An appropriate error estimate for T_2 is

$$\int_a^b f(x) dx - T_2 \approx \mathcal{E}(a, b) = \frac{1}{3}(T_2 - T_1).$$

The adaptive trapezoid algorithm then becomes:

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function ADAPTIVE-TRAPEZOID( $f, a, b, tol$ )
     $T_1 \leftarrow T(a, b)$ 
     $c \leftarrow (a + b)/2$ 
     $T_2 \leftarrow T(a, c) + T(c, b)$ 
     $\mathcal{E} \leftarrow (T_2 - T_1)/3$ 
    if  $|\mathcal{E}| \leq tol$  then
        return  $T_2$ 
    else
         $T_l \leftarrow$  ADAPTIVE-TRAPEZOID( $f, a, c, tol/2$ )
         $T_r \leftarrow$  ADAPTIVE-TRAPEZOID( $f, c, b, tol/2$ )
        return  $T_l + T_r$ 
    end if
end function

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Applied to the integral in the text, the algorithm gives:

| | | | |
|---|------------------------------------|--|------------------------------------|
| $tol = 2 \cdot 10^{-3}, a = 0.0, b = 0.8$ | | | |
| $T_1 = 0.23888, T_2 = 0.18317, \mathcal{E} = -1.86 \cdot 10^{-2}$ | | | |
| $tol = 1 \cdot 10^{-3} a = 0.0, b = 0.4$ | | $tol = 1 \cdot 10^{-3} a = 0.4, b = 0.8$ | |
| $T_1 = 0.031864, T_2 = 0.0239330$ | | $T_1 = 0.15130, T_2 = 0.14611$ | |
| $\mathcal{E} = -2.6 \cdot 10^{-3}$ | | $\mathcal{E} = -1.7 \cdot 10^{-3}$ | |
| $tol = 5 \cdot 10^{-4}$ | $tol = 5 \cdot 10^{-4}$ | $tol = 5 \cdot 10^{-4}$ | $tol = 5 \cdot 10^{-4}$ |
| $a = 0.0, b = 0.2$ | $a = 0.2, b = 0.4$ | $a = 0.4, b = 0.6$ | $a = 0.6, b = 0.8$ |
| $T_1 = 0.004000$ | $T_1 = 0.019931$ | $T_1 = 0.051159$ | $T_1 = 0.094947$ |
| $T_2 = 0.003000$ | $T_2 = 0.018953$ | $T_2 = 0.050320$ | $T_2 = 0.094536$ |
| $\mathcal{E} = -3.3 \cdot 10^{-4}$ | $\mathcal{E} = -3.3 \cdot 10^{-4}$ | $\mathcal{E} = -2.8 \cdot 10^{-4}$ | $\mathcal{E} = -1.4 \cdot 10^{-4}$ |

So

$$T = 0.003 + 0.018953 + 0.050320 + 0.094536 = 0.1668.$$

Task 4

a)

$$\int_{-1}^1 \frac{e^x}{\sqrt{1-x^2}} dx \approx \frac{e^{-1/\sqrt{3}}}{\sqrt{1-1/3}} + \frac{e^{1/\sqrt{3}}}{\sqrt{1-1/3}} = \sqrt{6} \cosh(1/\sqrt{3}) \approx 2.8692$$

b) Use the inner product $\langle f, g \rangle = \int_{-1}^1 (f(x)g(x)/\sqrt{1-x^2}) dx$, with Chebyshev polynomials as orthogonal polynomials. We choose $n = 1$, $T_2(x) = 2x^2 - 1$, i.e. $x_0 = -1/\sqrt{2}$, $x_1 = 1/\sqrt{2}$. The weights become

$$A_0 = \int_{-1}^1 \frac{(x - 1/\sqrt{2})}{(-1/\sqrt{2} - 1/\sqrt{2})} \cdot \frac{1}{\sqrt{1-x^2}} dx = \frac{\pi}{2},$$

$$A_1 = \int_{-1}^1 \frac{(x + 1/\sqrt{2})}{(1/\sqrt{2} + 1/\sqrt{2})} \cdot \frac{1}{\sqrt{1-x^2}} dx = \frac{\pi}{2}.$$

The approximation becomes $\frac{\pi}{2}(e^{-1/\sqrt{2}} + e^{1/\sqrt{2}}) \approx 3.9603$ which is significantly better than the answer in **a**).

c) The error is given by

$$E = K \cdot f^{(4)}(\nu), \quad K = \frac{1}{4!} \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \left(x^2 - \frac{1}{2}\right)^2 dx = \frac{\pi}{192}.$$

Since $|e^x| < e$ on $(-1, 1)$, we must have $|E| < e\pi/192 \approx 0.044$. The measured error is 0.017.

Task 5

The recursion formula from the note gives

$$\begin{aligned} \phi_0(x) &= 1, & \langle \phi_0, \phi_0 \rangle &= \int_0^\infty e^{-x} dx = 1 \\ & & \langle x\phi_0, \phi_0 \rangle &= \int_0^\infty e^{-x} x dx = 1 & \Rightarrow B_1 = 1 \\ \phi_1(x) &= x - 1, & \langle \phi_1, \phi_1 \rangle &= \int_0^\infty e^{-x} (x-1)^2 dx = 1 \\ & & \langle x\phi_1, \phi_1 \rangle &= 3 & \Rightarrow B_2 = 3, C_2 = 1 \\ \phi_2(x) &= (x-3)\phi_1(x) - \phi_0(x) = x^2 - 4x + 3 - 1 = x^2 - 4x + 2 \end{aligned}$$

Task 6

We must show that

$$\int_{-1}^1 \left((x^2-1)^k\right)^{(k)} \left((x^2-1)^j\right)^{(j)} dx = 0 \quad \text{for all } j < k.$$

Partial integration, $\int_a^b u dv = uv|_a^b - \int_a^b v du$ with

$$u = \left((x^2-1)^j\right)^{(j)}, \quad dv = \left((x^2-1)^k\right)^{(k)} dx$$

applied to the integral above gives

$$\left(\left((x^2-1)^k\right)^{(k-1)} \left((x^2-1)^j\right)^{(j)}\right)\Big|_{-1}^1 - \int_{-1}^1 \left(\left((x^2-1)^k\right)^{(k-1)} \left((x^2-1)^j\right)^{(j+1)}\right) dx.$$

The first term is zero (see hint). Again, we apply partial integration to the integral. After having done this $j+1$ times, we end up with

$$(-1)^{j+1} \int_{-1}^1 \left(\left((x^2-1)^k\right)^{(k-j-1)} \left((x^2-1)^j\right)^{(2j+1)}\right) dx = 0$$

since $\left((x^2-1)^j\right)^{(2j+1)} = 0$.