TMA4215 NUMERICAL MATHEMATICS

Suggested solution, exercise 3

1a) The first thing we do is to find the cardinal polynomials based on 2, 3 or 4 points respectively. where we choose x_0 and x_1 for the first degree polynomial, x_0, x_1 and x_2 for the second degree polynomial, and so on. You are free to choose other points. We get

$$\begin{aligned}
\ell_0^1(x) &= \frac{x-0.9}{-0.9} = \frac{0.9-x}{0.9} \\
\ell_1^1(x) &= \frac{x}{0.9}
\end{aligned}$$

$$\begin{aligned}
\ell_0^2(x) &= \frac{(x-0.9)(x-0.6)}{-0.9\cdot -0.6} \approx 1.852\,(x-0.9)(x-0.6) \\
\ell_1^2(x) &= \frac{x\,(x-0.6)}{0.9\,(0.9-0.6)} \approx 3.704x\,(x-0.6) \\
\ell_2^2(x) &= \frac{x\,(x-0.9)}{0.6\,(0.6-0.9)} \approx -5.556x\,(x-0.9)
\end{aligned}$$

$$\begin{aligned}
\ell_0^3(x) &= \frac{(x-0.9)(x-0.6)(x-0.4)}{-0.9\cdot -0.6\cdot -0.4} \approx -4.630\,(x-0.9)(x-0.6)(x-0.4) \\
\ell_1^3(x) &= \frac{x\,(x-0.6)(x-0.4)}{0.9\,(0.9-0.6)(0.9-0.4)} \approx 7.407x\,(x-0.6)(x-0.4) \\
\ell_2^3(x) &= \frac{x\,(x-0.9)(x-0.4)}{0.6\,(0.6-0.9)(0.6-0.4)} \approx -27.78x\,(x-0.9)(x-0.4) \\
\ell_3^3(x) &= \frac{x\,(x-0.9)(x-0.6)}{0.4\,(0.4-0.9)(0.4-0.6)} = 25.00x\,(x-0.9)(x-0.6)\,.
\end{aligned}$$

This means that we can find our interpolation polynomials as

$$\begin{aligned}
p_1(x) &= \sum_{i=0}^{1} f(x_i)\,\ell_i^1(x) \approx 0.42044986x + 1 \\
p_2(x) &= \sum_{i=0}^{2} f(x_i)\,\ell_i^2(x) \approx -0.070228596x^2 + 0.48365560x + 1.00 \\
p_3(x) &= \sum_{i=0}^{3} f(x_i)\,\ell_i^3(x) \approx 0.024757323x^3 - 0.10736457x^2 + 0.49702455x + 1.00.
\end{aligned}$$

Which finally yields

$$\begin{aligned}
|p_1(0.45) - f(0.45)| &\approx 1.50\times 10^{-2} \\
|p_2(0.45) - f(0.45)| &\approx 7.36\times 10^{-4} \\
|p_3(0.45) - f(0.45)| &\approx 1.63\times 10^{-5}
\end{aligned}$$

1b) Apply Theorem 6.2, pp.183-184. Since the interpolation abscissa and the point $x = 0.45$ all lie between 0 and 0.9, we have

$$|f(x) - p_n(x)| \leq \frac{\max_{\xi \in (0,0.9)} |f^{(n+1)}(\xi)|}{(n+1)!} \prod_{k=1}^{n} |x - x_k|$$

Additionally, we have

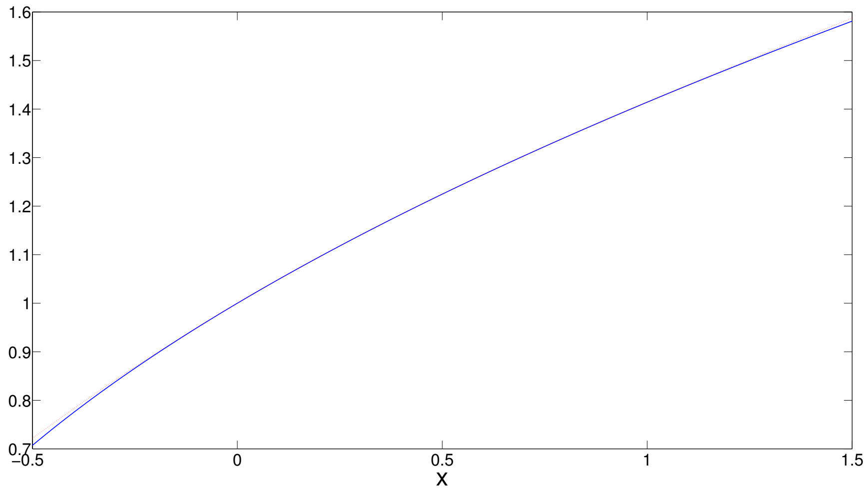
$$f' = \frac{1}{2\sqrt{1+x}},\ f'' = -\frac{1}{4(\sqrt{x+1})^3},\ f''' = \frac{3}{8(\sqrt{1+x})^5},\ f^{(4)} = -\frac{15}{16(\sqrt{1+x})^7}$$

which means that $\max_{\xi \in (0,0.9)} |f^{(n+1)}(\xi)| = |f^{(n+1)}(0)|$. We end up with the bounds

$$\begin{aligned}
|f(0.45) - p_1(0.45)| &\leq \frac{1/4}{2!} \cdot 0.45|0.45 - 0.9| \approx 2.53 \cdot 10^{-2} \\
|f(0.45) - p_2(0.45)| &\leq \frac{3/8}{3!} \cdot 0.45|0.45 - 0.9||0.45 - 0.6| \approx 1.90 \cdot 10^{-3} \\
|f(0.45) - p_3(0.45)| &\leq \frac{15/16}{4!} \cdot 0.45|0.45 - 0.9||0.45 - 0.6||0.45 - 0.4| \approx 5.93 \cdot 10^{-5}.
\end{aligned}$$

We see that all of the error bounds are rather conservative compared to the exact error.

1c) From Figure we see that if we increase the interval from $[0, 0.9]$ to $[-0.5, 1.5]$, the further we get away from our original interval, the larger the interval gets.



Figur 1: Solid: $f(x)$. Dotted: $p_3(x)$

2) By evaluating the polynomials p and q in the points given in the table, we see that they give the same result as the tabulated values of $f(x)$ This does not contradict the uniqueness statement in Theorem 6.1, because the theorem demands the degree of the interpolation polynomial to be less than the number of data points. Thus, the degree of q is too large to be covered by the theorem

3) We split the product in three parts and consider these separately, as recommended in the exercise text. First, we consider the cases $k < j$ and $k > j + 1$

$$
\begin{aligned}
k < j \quad &\Rightarrow \quad |x - x_k| < x_{j+1} - x_k = (j + 1 - k)h \\
k > j + 1 \quad &\Rightarrow \quad |x - x_k| < x_k - x_j = (k - j)h
\end{aligned}
$$

Now let $0 \leq \alpha \leq 1$ be such that $\alpha h = x - x_j$. This gives $|x - x_j||x - x_{j+1}| = \alpha(1 - \alpha)h^2$, and since $\max_{0 \leq \alpha \leq 1} = \alpha(1 - \alpha)$, the third case becomes

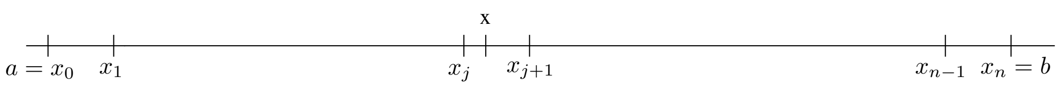
$$
|x - x_j||x - x_{j+1}| = \frac{1}{4}h^2
$$

Thus, we get

$$
\begin{aligned}
\prod_{k=0}^{n} |x - x_k| &\leq \frac{1}{4}h^2 \prod_{k=0}^{j-1} (j + 1 - k)h \cdot \prod_{k=j+2}^{n} (k - j)h \\
&= \frac{1}{4}h^{n+1}(j + 1)!(n - j)!
\end{aligned}
$$

This obtains its largest value for $j = 0$ or $j = n - 1$

$$
\prod_{k=0}^{n} |x - x_k| \leq \frac{1}{4}h^{n+1}n!.
$$



Figur 2: Figure illustrating the setting in task 2.

Together with Theorem 6.2 in S&M, this gives the error bound (1) in the text.

4a) The statement is true for $m = 0$. The rest will be proved by induction on m: Assume it is true for some $m \geq 0$.

$$
\begin{aligned}
f^{(m+1)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x} f^{(m)}(x) = \frac{\mathrm{d}}{\mathrm{d}x} 2^{\frac{m}{2}} e^x \sin\left(x + m\frac{\pi}{4}\right) \\
&= 2^{\frac{m}{2}} e^x \left(\sin\left(x + m\frac{\pi}{4}\right) + \cos\left(x + m\frac{\pi}{4}\right)\right) \qquad (1)
\end{aligned}
$$

and want to show that

$$
f^{(m+1)}(x) = 2^{\frac{m+1}{2}} e^x \sin\left(x + (m + 1)\frac{\pi}{4}\right) \qquad (2)
$$

We transform (2) to (1).

$$
\begin{aligned}
f^{(m+1)}(x) &= 2^{\frac{m+1}{2}} e^x \sin\left(x + (m + 1)\frac{\pi}{4}\right) \\
&= 2^{\frac{m}{2}} e^x \left(\sin\left(x + m\frac{\pi}{4}\right) \cos\left(\frac{\pi}{4}\right) + \cos\left(x + m\frac{\pi}{4}\right) \sin\left(\frac{\pi}{4}\right)\right) \\
&= 2^{\frac{m+1}{2}} \frac{1}{2}\sqrt{2} e^x \left(\sin\left(x + m\frac{\pi}{4}\right) + \cos\left(m + \frac{\pi}{4}\right)\right) \\
&= 2^{\frac{m}{2}} e^x \left(\sin\left(x + m\frac{\pi}{4}\right) + \cos\left(x + m\frac{\pi}{4}\right)\right)
\end{aligned}
$$

4b) From the exercise text we have that

$$M = \max_{x\in[-3,1]} \left| 2^{\frac{n+1}{2}} e^x \sin\left(x + \frac{(n+1)\pi}{4}\right) \right| \le 2^{\frac{n+1}{2}} e.$$

This means that

$$|f(x) - p_n(x)| \le \frac{1}{4(n+1)} 2^{\frac{n+1}{2}} e \left(\frac{4}{n}\right)^{n+1} = \frac{2^{\frac{5n+1}{2}} e}{(n+1)\, n^{n+1}} = s_n$$

and

n	8	9	10	11
s_n	$3.33 \cdot 10^{-3}$	$6.54 \cdot 10^{-4}$	$1.17 \cdot 10^{-4}$	$1.94 \cdot 10^{-5}$

so $n = 11$ seems sufficient.

5) By using Matlab we find the estimate $123.768 \cdot 10^6 \mathrm{Sm}^3$ for 1992. Furthermore, we find via the interpolation polynomial that we can expect $264.013 \cdot 10^6 \mathrm{Sm}^3$ in 2012 and $473.546 \cdot 10^6 \mathrm{Sm}^3$ in 2013. This doesn't sound very likely. The morale is that one shouldn't rely to much on an interpolation polynomial when doing extrapolation.