Given an equation

f(x) = 0

Reformulate the equation

x = g(x)

Fixed point iterations: Given x_0 ,

 $x_{k+1} = g(x_k), \quad k = 0, 1, 2, \dots$

The solution ξ of x = g(x) is called the *fixed point* of *g*. Example:

 $x \cdot \sin(x) - 1 = 0$

$$x = \frac{1}{\sin(x)}$$

$$x_{k+1} = \frac{1}{\sin(x_k)}$$

$$x_0 = 1.0$$

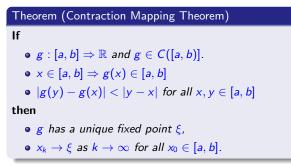
$$x_1 = \frac{1}{\sin(1.0)} = 1.1884$$

$$x_2 = \frac{1}{\sin(1.1884)} = 1.0779$$

Given the formulation

x = g(x)

and the corresponding iteration scheme.



NB! In this case, ξ is a *stable* fixed point.

Definition

Lipschitz continuity

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A function g is Lipschitz continuous on [a, b] if there is a positive constant L so that
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 $|g(x) - g(y)| \le L \cdot |x - y|$, for all $x, y \in [a, b]$.

Contraction

The function g is a contraction on [a, b] if L < 1.

• If $g \in C^1([a, b])$, then g is Lipschitz continuous with

 $L = \max_{x \in [a,b]} |g'(x)|.$

Definition

Let the sequence (x_k) converge to ξ , and let the error be

$$e_k = x_k - \xi.$$

If there exist a number $q \geq 1$ and a positive constant μ such that

$$\lim_{k \to \infty} \frac{|e_{k+1}|}{|e_k|^q} = \mu$$

then q is called the order of convergence and μ is called the asymptotic error constant.

- The higher *q*, the faster convergence.
- If q = 1, and μ < 1 then the convergence is linear, and μ is called the *rate* of convergence.

Theorem

Let ξ be a fixed point of g, and assume that $g \in C^q$ around ξ and satisfies

$$g^{(p)}(\xi) = 0, \quad p = 1, 2, \dots, q-1, \qquad g^{(q)}(\xi) \neq 0$$

for $q \ge 2$. Then the sequence generated by the fixed point iterations scheme satisfies

$$\lim_{s \to \infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|^q} = \frac{|g'(\xi)|}{q!}$$
(*)

for starting values x_0 sufficiently close to ξ .

Proof:

$$\begin{aligned} x_{k+1} - \xi &= g(x_k) - g(\xi) = g(\xi + (x_k - \xi)) - g(\xi) \\ \stackrel{\text{Taylor}}{=} g'(\xi)(x_k - \xi) + \dots + \frac{1}{(q-1)!}g^{(q-1)}(\xi)(x_k - \xi)^{(q-1)} + \frac{1}{q!}g^{(q)}(\eta_k)(x_k - \xi)^q \\ &= \frac{1}{q!}g^{(q)}(\eta_k)(x_k - \xi)^q \quad \text{proving (*).} \end{aligned}$$

Let $|g^{(q)}(x)| < M$ around ξ . (*M* exist because $g \in C^{(q)}$). Then

$$|x_1 - \xi| \le M |x_0 - \xi|^q = (M |x_0 - \xi|^{q-1}) |x_0 - \xi|.$$

Choose x_0 so that $M|x_0 - \xi|^{q-1}| < 1$. Then $|x_1 - \xi| < |x_0 - \xi|$.

Repeating the argument proves convergence for such x_0 .