Fixed point iterations: Idea

Given an equation

$$f(x) = 0$$

Reformulate the equation

$$x = g(x)$$

Fixed point iterations: Given x_0,

$$x_{k+1} = g(x_k), \quad k = 0, 1, 2, \ldots$$

The solution ξ of $x = g(x)$ is called the *fixed point* of g.

Example:

$$x \cdot \sin(x) - 1 = 0$$

$$x = \frac{1}{\sin(x)}$$

$$x_{k+1} = \frac{1}{\sin(x_k)}$$

$x_0 = 1.0$

$x_1 = \frac{1}{\sin(1.0)} = 1.1884$

$x_2 = \frac{1}{\sin(1.1884)} = 1.0779$

Contraction Mapping Theorem

Given the formulation

$$x = g(x)$$

and the corresponding iteration scheme.

Theorem (Contraction Mapping Theorem)

If

- $g : [a, b] \Rightarrow \mathbb{R}$ *and* $g \in C([a, b])$.
- $x \in [a, b] \Rightarrow g(x) \in [a, b]$
- $|g(y) - g(x)| < |y - x|$ *for all* $x, y \in [a, b]$

then

- g *has a unique fixed point* ξ,
- $x_k \to \xi$ *as* $k \to \infty$ *for all* $x_0 \in [a, b]$.

NB! In this case, ξ is a *stable* fixed point.

Definition

- **Lipschitz continuity**
  A function g is Lipschitz continuous on $[a, b]$ if there is a positive constant L so that

$$|g(x) - g(y)| \leq L \cdot |x - y|, \qquad \text{for all} \quad x, y \in [a, b].$$

- **Contraction**
  The function g is a contraction on $[a, b]$ if $L < 1$.

- If $g \in C^1([a, b])$, then g is Lipschitz continuous with

$$L = \max_{x \in [a,b]} |g'(x)|.$$

Rate of convergence

Definition

Let the sequence (x_k) converge to ξ, and let the error be

$$e_k = x_k - \xi.$$

If there exist a number $q \geq 1$ and a positive constant μ such that

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|^q} = \mu$$

then q is called the *order of convergence* and μ is called the *asymptotic error constant*.

- The higher q, the faster convergence.
- If $q = 1$, and $\mu < 1$ then the convergence is linear, and μ is called the *rate of convergence*.

High order schemes

Theorem

Let ξ be a fixed point of g, and assume that $g \in C^q$ around ξ and satisfies

$$g^{(p)}(\xi) = 0, \quad p = 1, 2, \ldots, q-1, \qquad g^{(q)}(\xi) \neq 0$$

for $q \geq 2$. Then the sequence generated by the fixed point iterations scheme satisfies

$$\lim_{k\to\infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|^q} = \frac{|g'(\xi)|}{q!} \tag{*}$$

for starting values x_0 sufficiently close to ξ.

Proof:

$$\begin{aligned}
x_{k+1} - \xi &= g(x_k) - g(\xi) = g(\xi + (x_k - \xi)) - g(\xi) \\
&\overset{\text{Taylor}}{=} g'(\xi)(x_k - \xi) + \cdots + \frac{1}{(q-1)!} g^{(q-1)}(\xi)(x_k - \xi)^{(q-1)} + \frac{1}{q!} g^{(q)}(\eta_k)(x_k - \xi)^q \\
&= \frac{1}{q!} g^{(q)}(\eta_k)(x_k - \xi)^q \qquad \text{proving } (*).
\end{aligned}$$

Let $|g^{(q)}(x)| < M$ around ξ. (M exist because $g \in C^{(q)}$). Then

$$|x_1 - \xi| \leq M|x_0 - \xi|^q = (M|x_0 - \xi|^{q-1})|x_0 - \xi|.$$

Choose x_0 so that $M|x_0 - \xi|^{q-1}| < 1$. Then $|x_1 - \xi| < |x_0 - \xi|$.

Repeating the argument proves convergence for such x_0.