NTNU – Trondheim
Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4215 Numerical Mathematics**

Academic contact during examination: Lars Hov Odsæter
Phone: +47 476 68 052

Examination date: 9. December 2015
Examination time (from–to): 15:00–19:00
Permitted examination support material: C:

- Endre Süli and David Mayers *An Introduction to Numerical Analysis* (a printout/copy is accepted)
- *TMA4215 Numerical Mathematics: Collection of Lecture Notes* (15. November 2013, 62 pages)
- Rottmann: *Matematisk formelsamling*
- Approved calculator

Language: English
Number of pages: 2
Number pages enclosed: 0

Checked by:

Date Signature

Problem 1

a) Given the following matrix A:

$$\begin{pmatrix} 3 & 1 & 2 \\ 6 & 3 & 3 \\ \text{-3} & 2 & \text{-3} \end{pmatrix} \tag{1}$$

Find a LU-factorization, i.e. find a lower triangular matrix L and an upper triangular matrix U such that $A = LU$.

b) Explain why the following matrix have no LU-decomposition:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

Problem 2

a) The value of a function $f(x)$ is known in the points:

x_i	-1	0	1
$f(x_i)$	0	-1	0

Find the polynomial $p(x)$ of lowest possible order interpolating $f(x)$ in these three points.

b) Find the polynomial $q(x)$ of lowest possible order which satisfies the interpolation conditions above *in addition* to the condition $f(-1/2) = -3/8$.

c) Construct a *near-minimax* interpolation polynomial $r(x) \in \mathcal{P}_2$, where $\mathcal{P}_2$ is the space of polynomials of maximum degree 2, for the function $g(x) = x^5$, on the interval $[-1, 1]$. Tabulate the difference $g(x) - r(x)$ using intervals $\Delta x = 0.2$ and plot the results.

d) Find a lower and an upper bound for the maximum error $||g(x) - r^*(x)||_\infty$ on the interval $[-1, 1]$, where $r^*(x) \in \mathcal{P}_2$ is the *true minimax* interpolation polynomial of $g(x)$, using the *near-minimax* interpolation polynomial $r(x)$ found above.

e) Assume that one is ask to approximate $e^{|x|}$ on the interval $[-1, 1]$ using either global or piecewise polynomials. What will you choose and why?

Problem 3

a) Determine a quadrature formula of the form

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=0}^{n} A_i f(x_i) \tag{3}$$

with $n = 1$ that is exact for all polynomials of degree 3.

b) Let $f(x) = x^4$ and compute the integral $\int_{-1}^{1} f(x)dx$ using the quadrature found above. Compare the found numerical result with the exact solution.

c) What is the important feature with Gaussian quadrature formulas (contrary to Newton-Cotes formulas) that guarantees convergence to exact solution when $n \to \infty$ for smooth functions $f(x)$?

d) Explain the relation between orthogonal polynomials and Gaussian quadrature formulas.

Problem 4

Given a scalar ordinary differential equation (ODE)

$$y' = f(t, y), \quad t_0 \leq t \leq t_{\text{end}}, \quad y(t_0) = y_0 \tag{4}$$

in which the function f, the interval $[t_0, t_{\text{end}}]$ and the initial value y_0 is given.

a) Show that the linear multistep method defined by

$$3y_{n+2} - 4y_{n+1} + y_n = 2hf_{n+2} \tag{5}$$

is consistent. What is *the order p* for this method?

b) Is this linear multistep method convergent?

c) Is the Runge-Kutta method given by this Butcher tableau A-stable?

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array} \tag{6}$$