SOLUTION TO THE EXAM IN
TMA4215 NUMERICAL MATHEMATICS

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Problem 1

a) We find an LU-factorization by Gauss elimination of A:

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 6 & 3 & 3 \\ -3 & 2 & -3 \end{pmatrix} \sim \begin{pmatrix} 3 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 3 & -1 \end{pmatrix} \sim \begin{pmatrix} 3 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} = U.$$

In each step we fill in the off-diagonal part of L with the coefficients used to eliminate the lower triangular part of A:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix}.$$

One can now verify that $A = LU$.

Note that the LU-factorization is not unique. Another possible factorization is

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{7} & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 6 & 3 & 3 \\ 0 & \frac{7}{2} & -\frac{3}{2} \\ 0 & 0 & \frac{2}{7} \end{pmatrix}.$$

b) Let $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. One may use different arguments to show that B has no LU-factorization:

- It is not possible to reduce B to an upper diagonal matrix by Gauss elimination without interchanging the rows (permutation).
- The leading principal matrix of order 1 is (0), and thus singular.
- The general form of L and U is

  $$L = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}, \qquad U = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$$

  Multiplication gives

  $$LU = \begin{pmatrix} ad & ae \\ bd & be + cf \end{pmatrix}$$

  It is not possible to have $ad = 0$, $ae = 1$ and $bd = 1$ simultaneously.

Problem 2

a) There are 3 interpolation points. This means that $p(x)$ is of maximum order 2. This task can be solved by different methods, e.g., Lagrange interpolation formula or divided differences. We will use the latter since we are asked to add one interpolation point in exercise b).

The divided difference table for this problem is:

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$
-1	0		
		-1	
0	-1		1
		1	
1	0		

From this we get

$$p(x) = 0 - (x + 1) + x(x + 1) = x^2 - 1.$$

b) We add the extra interpolation point to the divided difference table from a):

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
-1	0			
		-1		
0	-1		1	
		1		1
1	0		$\frac{3}{2}$	
		$\frac{1}{4}$		
$-\frac{1}{2}$	$-\frac{3}{8}$			

This gives

$$q(x) = p(x) + (x - 1)x(x + 1) = x^3 + x^2 - x - 1.$$

c) A *near-minimax* interpolation polynomial is defined on page 245 in S&M[1]. Thus we want $r(x) \in \mathcal{P}_2$ to be such that $g(x) - r(x)$ changes sign at 3 internal points on the interval $[-1, 1]$. By using the error formula (8.9) in S&M, we know that

$$g(x) - r(x) = \frac{g^{(3)}(\eta)}{3!}(x - \xi_0)(x - \xi_1)(x - \xi_2), \qquad \eta \in (-1, 1)$$

where ξ_i are the interpolation points. Since $g^{(3)}(x) = 60x^2 \geq 0$, and if we choose interpolation points away from the endpoints -1 and 1, the error $g(x) - r(x)$ will change sign 3 times.

[1] E. Süli and D. Mayers, *An introduction to Numerical Analysis*, Cambridge University Press (2003)

We choose to use Chebyshev nodes as the interpolation points,

$$\xi_j = \cos \frac{\left(j+\frac{1}{2}\right)\pi}{3}, \qquad j = 0, 1, 2,$$

i.e., $\xi_0 = -\frac{\sqrt{3}}{2},\ \xi_1 = 0,\ \xi_2 = \frac{\sqrt{3}}{2}$. The function values at these points are

$$g(\xi_0) = -\left(\frac{\sqrt{3}}{2}\right)^5, \quad g(\xi_1) = 0, \quad g(\xi_2) = \left(\frac{\sqrt{3}}{2}\right)^5.$$

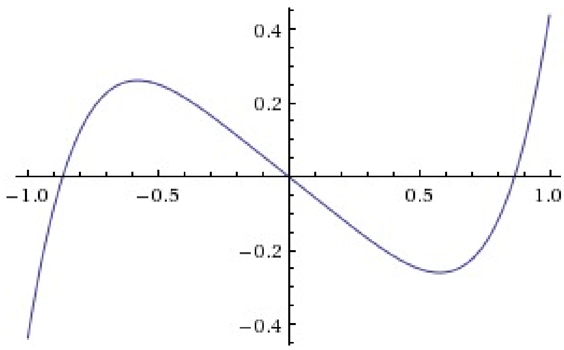
The near-minimax polynomial $r(x)$ is now the interpolation polynomial of these points. We observe that the 3 points lies on a straight line, so that $r(x)$ actually is a polynomial of degree 1. Searching for $r(x) = ax + b$ reveals that

$$r(x) = \frac{9}{16}x.$$

It is of course possible to determine $r(x)$ also by Lagrange interpolation or divided differences.

Next, we are asked to tabulate the difference $g(x) - r(x)$ and plot it:

x	$g(x) - r(x)$
-1.0	-0.43750
-0.8	0.12232
-0.6	0.25974
-0.4	0.21476
-0.2	0.11218
0.0	0.00000
0.2	-0.11218
0.4	-0.21476
0.6	-0.25974
0.8	-0.12232
1.0	0.43750



d) We use De la Vallée Poussin's Theorem (Theorem 8.3 on page 232 in S&M) with $n = 2$ and $f = g$. Furthermore, let x_i be the critical points of $g(x) - r(x)$, i.e., the local and global extremal points. From the plot in c) we see that there are $n+2 = 4$ such points (including the endpoints) and $g(x_i) - r(x_i)$ have alternating sign. The theorem states that the *true minimax* interpolation polynomial r^* satisfies the following error bound:

$$\|g - r^*\|_\infty = \min_{q \in \mathcal{P}_2} \|g - q\|_\infty \geq \min_{i=0,1,2,3} |g(x_i) - r(x_i)|$$

Furthermore, since r^* is the true minimax interpolation polynomial and since x_i are the extremal points of $g(x) - r(x)$, the following hold:

$$\|g - r^*\|_\infty = \min_{q \in \mathcal{P}_2} \|g - q\|_\infty \leq \|g - r\|_\infty = \max_{i=0,1,2,3} |g(x_i) - r(x_i)|$$

To sum up, we have the following bounds for $\|g - r^*\|_\infty$:

$$\min_{i=0,1,2,3} |g(x_i) - r(x_i)| \leq \|g - r^*\|_\infty \leq \max_{i=0,1,2,3} |g(x_i) - r(x_i)|.$$

By differentiating $g(x) - r(x)$ and put it equal to zero, the extremal points are

$$x_0 = -1,\ x_1 = -\sqrt[4]{\frac{9}{80}},\ x_2 = \sqrt[4]{\frac{9}{80}},\ x_3 = 1.$$

From the plot or by insertion, we see that

$$\min_{i=0,1,2,3} |g(x_i) - r(x_i)| = |g(x_1) - r(x_1)| = |g(x_2) - r(x_2)| \approx 0.26062,$$
$$\max_{i=0,1,2,3} |g(x_i) - r(x_i)| = |g(x_0) - r(x_0)| = |g(x_3) - r(x_3)| \approx 0.43750.$$

Concluding this task, we have that

$$0.26062 \lessapprox \|g - r^*\|_\infty \lessapprox 0.43750.$$

OBS! Task 2c) and d) can be solved by different approaches, and the near-minimax interpolation polynomial r(x) is not unique.

e) *OBS! This task was removed from the exam since it was not included in the Norwegian version. We still include a brief answer here.*

The function $\mathrm{e}^{|x|}$ is not continuously differentiable in $x = 0$. A global polynomial approximation would be smooth (all derivatives are continuous) on the whole interval. To allow for approximations whose derivative is not continuous in $x = 0$, one should use piecewise polynomials and choose $x = 0$ as one of the endpoints for the subintervals.

Problem 3

a) We know that Gauss-Legendre quadrature of order $n = 1$ is exact for polynomials of degree $2n+1 = 3$ (Theorem 10.1 on page 282 in S&M). The Legendre polynomial of degree 3 is $\varphi_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$. The quadrature nodes are the roots of this polynomial, i.e.,

$$x_0 = -\frac{1}{\sqrt{3}}, \quad x_1 = \frac{1}{\sqrt{3}}.$$

Furthermore, the weights A_i are given as

$$A_i = \int_{-1}^{1} [L_i(x)]^2 \, dx, \qquad L_i(x) = \prod_{j=0, j\neq i}^{n=1} \frac{x - x_j}{x_i - x_j}.$$

With our quadrature nodes,

$$L_0(x) = -\frac{\sqrt{3}}{2}\left(x - \frac{1}{\sqrt{3}}\right), \qquad L_1(x) = \frac{\sqrt{3}}{2}\left(x + \frac{1}{\sqrt{3}}\right),$$

so that

$$\begin{aligned} A_0 &= \frac{3}{4}\int_{-1}^{1}\left(x - \frac{1}{\sqrt{3}}\right)^2 dx = 1, \\ A_1 &= \frac{3}{4}\int_{-1}^{1}\left(x + \frac{1}{\sqrt{3}}\right)^2 dx = 1. \end{aligned}$$

Hence, the quadrature formula reads

$$\int_{-1}^{1} f(x)\,dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$$

One can verify that it is exact for polynomials of degree 3.

Alternative approach: One may also use the method of *direct construction* as described in Section 10.3 in S&M. Thus, we find the unknowns A_0, A_1, x_0 and x_1 such that the quadrature is exact for $f(x) = x^i$ for $i = 0, 1, 2, 3$. This gives the four equations

$$\begin{aligned} A_0 + A_1 &= \int_{-1}^{1} dx = 2, \\ A_0x_0 + A_1x_1 &= \int_{-1}^{1} x\,dx = 0, \\ A_0x_0^2 + A_1x_1^2 &= \int_{-1}^{1} x^2\,dx = \frac{2}{3}, \\ A_0x_0^3 + A_1x_1^3 &= \int_{-1}^{1} x^3\,dx = 0. \end{aligned}$$

This is a system of four non-linear equations with four unknowns, and the solution is $A_0 = A_1 = 1$, $x_0 = -\frac{1}{\sqrt{3}}$ and $x_1 = \frac{1}{\sqrt{3}}$.

b) With $f(x) = x^4$,

$$\int_{-1}^{1} f(x)\,dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = \left(-\frac{1}{\sqrt{3}}\right)^4 + \left(\frac{1}{\sqrt{3}}\right)^4 = \frac{2}{9}.$$

The exact value is $\int_{-1}^{1} f(x)\,dx = \frac{2}{5}$, so the error is $\frac{2}{5} - \frac{2}{9} = \frac{8}{45} \approx 0.1778$. It is expected that the error is not zero since f is a polynomial of degree 4.

c) For Gaussian quadrature all weights are positive, while this is not guaranteed for Newton-Cotes formulas.

The Gauss quadrature weights are gives as

$$W_k = \int_{-1}^{1} w(x)[L_k(x)]^2 \, dx,$$

where $w(x)$ is a positive weighting function. Clearly $W_k \geq 0 \, \forall k$.

d) Gaussian quadrature formulas are defined as the exact integration of some positive weight function $w(x)$ times the Hermite polynomials whose interpolation points are roots of orthogonal polynomials with respect to $w(x)$.

To be more precise, Gauss quadrature $\mathcal{G}_n(f)$ (on the interval $[-1, 1]$) is defined as

$$\int_{-1}^{1} w(x)f(x) \approx \mathcal{G}_n(f) = \sum_{k=0}^{n} W_k f(x_k),$$

where W_k is defined above. Let φ_i be the set of orthogonal polynomials wrt. $w(x)$, i.e.,

$$\int_{-1}^{1} \varphi_i(x)\varphi_j(x)w(x) \, dx = 0, \quad \text{for } i \neq j.$$

Then φ_{n+1} is a polynomial of degree $n+1$. Now, the quadrature points x_k are the roots of φ_{n+1} and $L_k(x)$ are defined from these roots as

$$L_k(x) = \prod_{i=0, i\neq k}^{n} \frac{x - x_i}{x_k - x_i}.$$

Problem 4

a) We calculate the coefficients C_q defined in Eq. (71) in the lecture notes[2]:

$$C_0 = \sum_{l=0}^{k} \alpha_l = 3 - 4 + 1 = 0,$$

$$C_1 = \frac{1}{1!}\sum_{l=0}^{k} (l\alpha_l - \beta_l) = 2 \cdot 3 + 1 \cdot (-4) - 2 = 0,$$

$$C_2 = \frac{1}{2!}\sum_{l=0}^{k} (l^2\alpha_l - 2l\beta_l) = \frac{1}{2}\,(4 \cdot 3 + 1 \cdot (-4) + 2 \cdot 2 \cdot 2) = 0,$$

$$C_3 = \frac{1}{3!}\sum_{l=0}^{k} (l^3\alpha_l - 3l^2\beta_l) = \frac{1}{6}\,(8 \cdot 3 + 1 \cdot (-4) - 3 \cdot 4 \cdot 2) = -\frac{4}{6}.$$

The method is consistent since $C_0 = C_1 = 0$ and of order 2 since $C_2 = 0$ but $C_3 \neq 0$.

[2] http://www.math.ntnu.no/emner/TMA4215/2014h/Exercises/AK-LectureNotes.pdf

b) Theorem 6.28 (Dahlquist) in the lecture notes states that convergence is equivalent to zero-stability plus consistency. We proved consistency in a). The method is zero-stable if the roots of the characteristic polynomial

$$\rho(r) = \sum_{l=0}^{k} \alpha_l r^l = 3r^2 - 4r + 1$$

has roots $r_i,\ i = 1, 2$, s.t.

(i) $|r_i| \leq 1, \quad i = 1, 2$

(ii) $|r_i| < 1$, if r_i is a multiple root.

The roots of $\rho(r)$ are $r_1 = 1$ and $r_2 = \frac{1}{3}$, so that both criteria are satisfied.

c) The Runge-Kutta method is explicit (all coefficients on and above the diagonal are zero), and thus can not be A-stable (see page 52 in the lecture notes).