Welcome to

TMA4215 Numerical Mathematics

https://wiki.math.ntnu.no/tma4215/2015h/start

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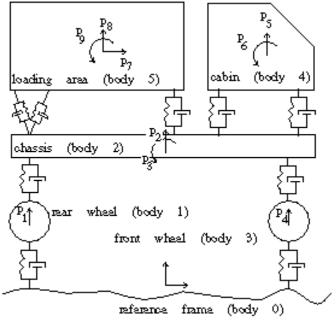
Why learn Numerical Mathematics?

- It is part of the curriculum.
- My supervisor told me to.
- It sounds interesting / fun / challenging.

  - The world of technology/science is described by mathematics (equations, often differential equations).
  - These can only be solved by numerical algorithms
    (a recipe to find an approximation to the solution using a computer).

Case: A Truck Model

Mathematical model:



Physics:
Newton's second law $F = ma$.

Rear wheel:

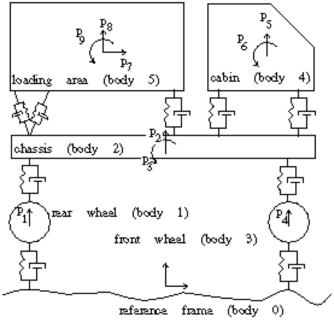
$$m_1 p_1'' = f_{10} + f_{12} - m_1 g$$

where e.g.:

$$f_{10} = k_{10}(p_1 - u(t)) + d_{01}(p_1' - u'(t)) + f_{10}^0$$

p_1	Vertical position
m_1	Mass of the
g	Gravitational constant
k_{01}, d_{01}	Stiffness and damping constants.
$u(t)$	From the road

Similar for $p_2, p_3, \ldots, p_9$.

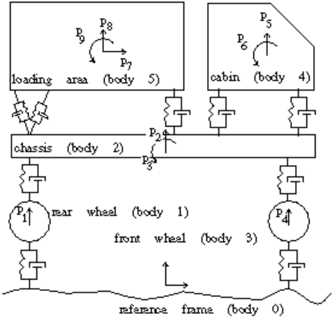


Putting everything together:

- We get a system of 9 second order differential equations

$$Mp'' = f(t, p, p'),$$

- which you will hardly solve by hand.
- In December, you will not only be able to solve this problem numerically
- but you have learned how to construct your own algorithms for doing so.



Objective of the course

Given a class of problems (linear or nonlinear equations, integrals, differential equations), you will learn how to

- develop a numerical algorithm
- implement it (computer program)
- do convergence/stability analysis
- do verification (testing, testing, testing)

The focus is on ideas and principles.

Your background

- Mathematics 1-4
- Some programming experience (MATLAB or Python)
- TMA4145 Linear methods this fall is not required, but will be an advantage.

Taylor's theorem

Theorem (A.4 and A.5)

If f is a $n+1$ times continuous function, then

$$f(x) = f(a) + (x-a)f'(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) + R_{n+1}(x)$$

where

$$R_{n+1}(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi(x)) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt$$

and $\xi(x)$ is somewhere between a and x.

Big O - notation (Ordo)

Definition

$$f(x) = \mathcal{O}(g(x)) \quad \text{as} \quad x \to a$$

if and only there exist positive numbers δ and C such that

$$|f(x)| \leq C|g(x)| \quad \text{for} \quad |x - a| < \delta$$

Common use:

$$f(x) = \mathcal{O}((x - a)^p)$$

for some fixed point a and some integer p.

Floating point model

Computers have finite memory, hence not every number can be represented exactly on a computer.
Examples: $\sqrt{2}$, π have a infinite number of digits.
To fit in a computer, real numbers are approximated by the *floating point model*:

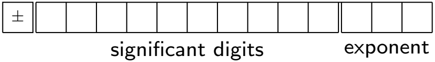
- Binary system is used: If $r \in \mathbb{R}$ then

$$\begin{aligned} r &= \pm(\alpha_k 2^k + \alpha_{k-1} 2^{k-1} + \cdots + \alpha_1 2 + \alpha_0 + \alpha_{-1} 2^{-1} + \cdots) \\ &= \pm(\alpha_k . \alpha_{k-1} \ \ldots \ \alpha_1 \, \alpha_0 \, \alpha_{-1} \ \ldots) \cdot 2^k \end{aligned}$$

  where $\alpha_i \in \{0, 1\}$, $\alpha_k \neq 0$.

- in a computer, a fixed amount of memory is allocated to represent a number:

$$fl(r) = \pm(\alpha_k . \alpha_{k-1} \ \ldots \ \alpha_{k-m-1} \, \tilde{\alpha}_{k-m}) \cdot 2^e$$



Floating point example

Let

$$
\begin{aligned}
r = \pi &= 3.141\ 592\ 653\ 589\ 793\ 238\ 462\ 643\ 383\ 279\ 502\ 884\ 197\ 169\ 399\ 375\ \ldots \\
&= 1 \cdot 2^{1} + 1 \cdot 2^{0} + 0 \cdot 2^{-1} + 0 \cdot 2^{-2} + 1 \cdot 2^{-3} + 0 \cdot 2^{-4} + 0 \cdot 2^{-5} + 1 \cdot 2^{-6} \\
&\quad + 0 \cdot 2^{-7} + 0 \cdot 2^{-8} + 0 \cdot 2^{-9} + 0 \cdot 2^{-10} + 1 \cdot 2^{-11} + 1 \cdot 2^{-12} + 1 \cdot 2^{-13} + \ldots \\
&= (1.1001001000011111101101010100010001000010110100011 00\ldots) \cdot 2^{1}
\end{aligned}
$$

On a computer (with 10 significant bits):

$$fl(r) = +1.100100100 \cdot 2^{1} = \mathbf{3.14}06$$

With 24 significant bits:

$$fl(r) = \mathbf{3.141592}7410125732421875$$

Floating point numbers

Double precision IEEE 745 (Matlab)

1 bit	52 bits	11 bits
	significant digits	exponent

- **Rounding:** $r \to fl(r)$.
- **Machine accuracy:** The smallest number ϵ such that

$$fl(1+\epsilon) \neq 1$$

- **Relative rounding error** δ_r:

$$fl(r) = r\,(1+\delta_r), \qquad |\delta_r| \leq \varepsilon.$$

- **Loss of significant digits:** loss of precision due subtraction of two almost equal numbers.