Solving non-linear equations

Trond Kvamsdal

Given an equation

f(x) = 0

## Reformulate the equation

x = g(x)

Fixed point iterations: Given  $x_0$ ,

 $x_{k+1} = g(x_k), \quad k = 0, 1, 2, \dots$ 

The solution  $\xi$  of x = g(x) is called the *fixed point* of *g*. Example:

 $x \cdot \sin(x) - 1 = 0$ 

$$x = \frac{1}{\sin(x)}$$

$$x_{k+1} = \frac{1}{\sin(x_k)}$$

$$x_0 = 1.0$$
  

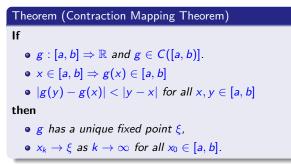
$$x_1 = \frac{1}{\sin(1.0)} = 1.1884$$
  

$$x_2 = \frac{1}{\sin(1.1884)} = 1.0779$$

Given the formulation

x = g(x)

and the corresponding iteration scheme.



NB! In this case,  $\xi$  is a *stable* fixed point.

# Definition

## Lipschitz continuity

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A function g is Lipschitz continuous on [a, b] if there is a positive constant L so that
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 $|g(x) - g(y)| \le L \cdot |x - y|$ , for all  $x, y \in [a, b]$ .

# Contraction

The function g is a contraction on [a, b] if L < 1.

• If  $g \in C^1([a, b])$ , then g is Lipschitz continuous with

 $L = \max_{x \in [a,b]} |g'(x)|.$ 

#### Definition

Let the sequence  $(x_k)$  converge to  $\xi$ , and let the error be

$$e_k = x_k - \xi.$$

If there exist a number  $q \geq 1$  and a positive constant  $\mu$  such that

$$\lim_{k \to \infty} \frac{|e_{k+1}|}{|e_k|^q} = \mu$$

then q is called the order of convergence and  $\mu$  is called the asymptotic error constant.

- The higher *q*, the faster convergence.
- If q = 1, and μ < 1 then the convergence is linear, and μ is called the *rate* of convergence.

#### Theorem

Let  $\xi$  be a fixed point of g, and assume that  $g \in C^q$  around  $\xi$  and satisfies

$$g^{(p)}(\xi) = 0, \quad p = 1, 2, ..., q - 1, \qquad g^{(q)}(\xi) \neq 0$$

for  $q \ge 2$ . Then the sequence generated by the fixed point iterations scheme satisfies

$$\lim_{s \to \infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|^q} = \frac{|g'(\xi)|}{q!}$$
(\*)

for starting values  $x_0$  sufficiently close to  $\xi$ .

Proof:

$$\begin{aligned} x_{k+1} - \xi &= g(x_k) - g(\xi) = g(\xi + (x_k - \xi)) - g(\xi) \\ &\stackrel{\text{Taylor}}{=} g'(\xi)(x_k - \xi) + \dots + \frac{1}{(q-1)!}g^{(q-1)}(\xi)(x_k - \xi)^{(q-1)} + \frac{1}{q!}g^{(q)}(\eta_k)(x_k - \xi)^q \\ &= \frac{1}{q!}g^{(q)}(\eta_k)(x_k - \xi)^q \quad \text{proving (*).} \end{aligned}$$

Let  $|g^{(q)}(x)| < M$  around  $\xi$ . (*M* exist because  $g \in C^{(q)}$ ). Then

$$|x_1 - \xi| \le M |x_0 - \xi|^q = (M |x_0 - \xi|^{q-1}) |x_0 - \xi|.$$

Choose  $x_0$  so that  $M|x_0 - \xi|^{q-1}| < 1$ . Then  $|x_1 - \xi| < |x_0 - \xi|$ . Repeating the argument proves convergence for such  $x_0$ .