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Examination paper for **TMA4215 Numerical Mathematics**

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Permitted examination support material: C:

- Endre Süli and David Mayers *An Introduction to Numerical Analysis* (a printout/copy is accepted)
- *TMA4215 Numerical Mathematics: Collection of Lecture Notes* (15. November 2013, 62 pages)
- Rottmann: *Matematisk formelsamling*
- Approved calculator

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Problem 1

a) The matrix is symmetric as $a_{ij} = a_{ji}\ \forall i, j = 1, 3$. Furthermore, all diagonal terms are positive and we have $a_{ii} > \sum_{j=1, j\neq i}^{3} |a_{ij}|$, i.e. the matrix is diagonally dominant. It then follows then from Gershgorin theorem that all the eigenvalues are real and positive, hence the matrix is positive definit.

b) The conditioning number $\kappa(A) = ||A||||A^{-1}||$. For matrices with large condition numbers (ill-condition equation system) may small perturbations in the right hand side result in large errors in the computed unknowns.

Problem 2

a) The interpolating polynom of lowest order is: $p_2 = 2 + 3x + x^2$. This may be found by using the definition of Lagrange polynomial (Equation (6.2) in Suli and Mayers), but here the use of Newton interpolation formula and divided differences (see slides by Elena Celledoni) may be consider advantageous as that makes next question easier to solve.

$$
\begin{array}{c|cccc}
x_i & f[x_i] & f[x_i, x_{i+1}] & f[x_i, x_{i+1}, x_{i+2}] \\
\hline
0.0 & 2.0 & & \\
 & & 5.0 & \\
2.0 & 12.0 & & 1.0 \\
 & & 8.0 & \\
3.0 & 20.0 & &
\end{array}
\tag{1}
$$

Thus we get: $p_2(x) = 2 + 5(x - 0) + 1(x - 0)(x - 2) = 2 + 3x + x^2$.

b) We now just add one row and column to our table:

$$
\begin{array}{c|ccccc}
x_i & f[x_i] & f[x_i, x_{i+1}] & f[x_i, x_{i+1}, x_{i+2}] & \\
\hline
0.0 & 2.0 & & & \\
 & & 5.0 & & \\
2.0 & 12.0 & & 1.0 & \\
 & & 8.0 & & 0.0 \\
3.0 & 20.0 & & 1.0 & \\
 & & 7.0 & & \\
1.0 & 6.0 & & &
\end{array}
\tag{2}
$$

Thus we get: $p_2(x) = 2+5(x-0)+1(x-0)(x-2)+0(x-0)(x-2)(x-3) = 2+3x+x^2$.

c) A third order polynomial $p_3(x) = c_0 + c_1x + c_2x^2c_3x^3$ has four unknown coefficient that may be determined by four independent constraints. Here we choose the following constraints: $p(0) = f(0), p'(0) = f'(0), p(a) = f(a), andp'(a) = f'(a)$, i.e for $f(x) = x^5$ we get the following system to solve:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & a & a^2 & a^3 \\ 0 & 1 & 2a & 3a^2 \end{bmatrix} \times \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ a^5 \\ 5a^4 \end{bmatrix} \tag{3}$$

We see immediately that $c_0 = c_1 = 0$ thus we end up with the following system:

$$\begin{bmatrix} a^2 & a^3 \\ 2a & 3a^2 \end{bmatrix} \times \begin{bmatrix} c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a^5 \\ 5a^4 \end{bmatrix} \tag{4}$$

This gives $c_2 = -2a^3$ and $c_3 = 3a^2$.

d) First note that $2n+1 = 3$ i.e. $n = 1$.
We have that $f(x) - p_3(x) = x^5 - 3a^2x^3 + 2a^3x^2$.
Theorem 6.4 on page 190 in the textbook by Süli and Mayers:
$\Pi_{n+1}(x) = (x-0)(x-a)$ and $f^{(2n+2)}(\xi) = 120\xi$

We then get: $\frac{120}{24}\xi(x^4 - 2ax^3 + a^2x^2)$, so if we insert $\xi = \frac{1}{5}(x+2a)$ we obtain $f(x) - p_3(x)$. Since $f^{(2n+2)}(x)$ is a linear function this value is unique.

Problem 3

a) Determine the values c_j, $j = -1, 0, 1, 2$, such that the quadrature rule $Q(f) = c_{-1}f(-1) + c_0f(0) + c_1f(1) + c_2f(2)$ gives the correct value for the integral $\int_0^1 f(x)dx$ when f is any polynomial of degree 3.

Direct construction gives as explained in Section 10.3 on page 280-281 in the textbook of Suli and Mayers:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 8 \end{bmatrix} \times \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix} \tag{5}$$

The solution is: $c_{-1} = -\frac{1}{24}, c_0 = \frac{13}{24}, c_1 = \frac{13}{24}$, and $c_2 = -\frac{1}{24}$. Notice that the weights c_i are symmetric about the midpoint 0.5 of the given interval and sums to 1.0

b) Define the composite trapezium rule T_m, the composite Simpson rule S_m and the composite midpoint rule M_m, each with m subintervals. Show that: $M_m = 2T_{2m} - T_m$, $S_m = (4T_{2m} - T_m)/3$, and $S_m = (2M_m + T_m)/3$

The composite trapezium rule:

$$T_m(a,b) = h\left[\frac{1}{2}f(x_0) + f(x_1) + \ldots + f(x_{m-1}) + \frac{1}{2}f(x_m)\right] \tag{6}$$

The composite Simpson rule:

$$\begin{aligned} S_m(a,b) = \frac{h}{3}[f(x_0) + 4f(\frac{1}{2}(x_0+x_1)) + 2f(x_1) + 4f(\frac{1}{2}(x_1+x_2)) + 2f(x_2) \\ + \ldots + 2f(x_{m-1}) + 4f(\frac{1}{2}(x_{m-1}+x_m)) + f(x_m)] \end{aligned} \tag{7}$$

Notice: We have m intervals that gives $2m+1$ points, i.e. to be consistent with the intervals for T_m and M_m ($h = (b-a)/m$ we use the midpoints $\frac{1}{2}(x_i - x_{i+1})$ for all $i = 0, m-1$.

The composite Midpoint rule:

$$M_m(a,b) = h\left[f(\frac{1}{2}(x_0+x_1)) + \ldots + f(\frac{1}{2}(x_{m-1}+x_m))\right] \tag{8}$$

To show that $M_m = 2T_{2m} - T_m$, first observe:

$$\begin{aligned} 2T_{2m}(a,b) = h[\frac{1}{2}f(x_0) + f(\frac{1}{2}(x_0+x_1)) + f(x_1) \\ + \ldots + f(\frac{1}{2}(x_{m-2}+x_{m-1})) + f(x_{m-1}) + \frac{1}{2}f(x_m)] \end{aligned} \tag{9}$$

When we subtract T_m from T_{2m} all the terms at the points x_i for $i = 0, m$ cancels out and only the the terms $f(\frac{1}{2}(x_i + x_{i+1}))$ for $i = 0, m-1$ at the midpoints remains, i.e. we get M_m.

We now show that $S_m = (2M_m + T_m)/3$:

$$S_m = \sum_{i=0}^{m-1} \frac{h}{6}\left[f(x_i) + 4f(\frac{1}{2}(x_i+x_{i+1})) + f(x_{i+1})\right] \tag{10}$$

We see now that the $2M_m/3$ corresponds to all the terms at the midpoints $\frac{1}{2}(x_i + x_{i+1}))$ for $i = 0, m-1$ and $T_m/3$ corresponds to all terms at the points x_i for $i = 0, m$.

Finally we easily may show that $S_m = (4T_{2m} - T_m)/3$ by insertion:

$$S_m = (2M_m + T_m)/3 = 2((2T_{2m} - T_m) + T_m)/3 = (4T_{2m} - T_m)/3 \tag{11}$$

c) We will avoid any numerical integration methods that are based on global Lagrange interpolation using equidistant interpolation points, i.e. Newton-Cotes quadrature as they have negative weights for high polynomial order. Gaussian quadrature will work fine. Composite Newton-Cotes may work depending on the choice of subintervals. Composite Gauss will be a good choice.

d) The characteristic of orthogonal polynomials is that given and interval $[a, b]$ and a weight function $w(x)$ we have:

$$\int_{-1}^{1} \phi_i \phi_j w(x) dx = 0 \ \ \forall i \neq j \tag{12}$$

See Definition 9.4 page 260 in the textbook of Süli and Mayers.

Orthogonal polynomials play an important role in finding the best polynomial approximation of function measured in the 2-norm. The zeroes of a set of orthogonal polynomials may also be used to determine the optimal numerical sampling points in a numerical integration scheme.

E.g. the Legendre polynomials are a set of orthogonal polynomials and on the interval $[-1, 1]$ they are: $\phi_0(x) = 1$, $\phi_1(x) = x$, and $\phi_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$. The orthogonality condition is

$$\int_{-1}^{1} \phi_i \phi_j dx = 0 \ \ \forall i \neq j \tag{13}$$

We have that

$$\int_{-1}^{1} 1 \cdot x dx = |_{-1}^{1} \frac{1}{2} x^2 = 0 \tag{14}$$

$$\int_{-1}^{1} 1 \cdot \frac{3}{2}x^2 - \frac{1}{2} dx = |_{-1}^{1} \frac{1}{2} x^3 - \frac{1}{2} x = 0 \tag{15}$$

$$\int_{-1}^{1} x \cdot \frac{3}{2}x^2 - \frac{1}{2} dx = |_{-1}^{1} \frac{3}{8} x^4 - \frac{1}{4} x^2 = 0 \tag{16}$$

Problem 4

a) The given one-step method is denoted *Improved Euler* and is a Runge-Kutta method with the following Butcher tableau:
In order to be consistent a method has to be at least of order 1, and for

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \tag{17}$$

Runge-Kutta methods that means

$$\sum_{i=1}^{s} b_1 = 1 \tag{18}$$

Improved Euler is a two-stage method, $s = 2$ and we have:

$$\sum_{i=1}^{2} b_1 + b_2 = \frac{1}{2} + \frac{1}{2} = 1 \tag{19}$$

i.e. Improved Euler is consistent.

Regarding the the local truncation error we observe that *Improved Euler* is of order $p = 2$ as

$$\sum_{i=1}^{s} b_i c_i = \frac{1}{2} \tag{20}$$

but i does not fulfill the 3rd order condition (see table on page 46 in The Collection of Lecture notes). Thus the local truncation error l_{n+1} is on the form:

$$l_{n+1} = \Psi(t_n, y_n)h^3 + \mathcal{O}(h^4) \tag{21}$$

For *Improved Euler* the local truncation error is given by:

$$l_{n+1} = \frac{1}{6}h^3\left[f_y(f_x + f_y f) - \frac{1}{2}(f_{xx} + 2f_{xy}f + f_{yy}f^2)\right] + \mathcal{O}(h^4) \tag{22}$$

b) To find the stability region we solve the eigenvalue problem

$$y' = \lambda y \tag{23}$$

with the *Improved Euler* method:

$$y_{n+1} = y_n + \frac{1}{2}h\left[\lambda y_n + \lambda(y_n + h\lambda y_n)\right] = \left(1 + h\lambda + \frac{1}{2}h^2\lambda^2\right) y_n \tag{24}$$

Let $z = h\lambda$, the stability function reads:

$$g(z) = 1 + z + \frac{1}{2}z^2 \tag{25}$$

The region of stability, G, is defined as that part of the complex plane where $g(z)$ is contracting:

$$G = \{z \in C \text{ such that } |g(z)| < 1\} \tag{26}$$

Improved Euler is an explicit Runge-Kutta method, i.e. it is not A-stable. To be A stable the region of stability has to contain the whole negative (left) complex half-plane, and this is not the case for the stability function g(z) for *Improved Euler* given above.

c) Linear multistep methods LMM may be written on the form:

$$\sum_{l=0}^{k} \alpha_l y_{n+1} = h \sum_{l=0}^{k} \beta_l f_{n+l} \tag{27}$$

Here, we have $\alpha_0 = -5$, $\alpha_1 = 4$, $\alpha_2 = 1$, $\beta_0 = 2$, $\beta_1 = 4$ and $k = 2$.

By inserting this in Equation (71) on Page 55 in the Collection of Lecture Notes we find: $C_0 = \ldots = C_3 = 0$ whereas $C_4 \neq 0$, i.e. the order of the method is 3.

d) A LMM is convergent if and only if it is both Consistent and Zero-stable. The given LMM is consistent (i.e. order $p \geq 1$). To be zero-stable all the roots r_i of the corresponding *first characteristic polynomial*:

$$\sum_{l=0}^{k} \alpha_l r^l \tag{28}$$

has to fulfill $|r_i| \leq 1$. We have:

$$g(r) = r^2 + 4r - 5 = (r - 1)(r + 5) \tag{29}$$

The roots are: $r_1 = 1$ and $r_2 = -5$, i.e. the given LMM is not convergent.