Department of Mathematical Sciences

Examination paper for **TMA4215 Numerical Mathematics**

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Examination date: 2. December 2013
Examination time (from–to): 09:00–13:00
Permitted examination support material: C:

- Endre Süli and David Mayers, *An Introduction to Numerical Analysis* (a printout/copy is accepted)
- TMA4215 Numerical Mathematics: Collection of lecture notes (62 pages)
- Rottmann: *Matematisk formelsamling*
- Approved calculator.

Language: English
Number of pages: 3
Number pages enclosed: 0

Checked by:

Date Signature

Problem 1 (50%)
In this exercise, you are given small questions from different parts of the curriculum. *Notice that yes/no answers without justification gives no credit.*

a) Set up the interpolation polynomial $p(x)$ *in Lagrange form* for the following data set:

x_i	1	3	4
y_i	3	1	8

b) The equation

$$2x = e^x - 1$$

has a solution at $x^* = 1.25643$. Will the iteration scheme

$$x_{k+1} = \frac{e^{x_k} - 1}{2}$$

converge to x^* for x_0 sufficiently close?

c) Find the degree of accuracy (or precision) for the quadrature formula $Q(f)$:

$$\int_0^1 f(x)dx \approx Q(f) = \frac{3\,f\left(\frac{1}{3}\right) + f(1)}{4}.$$

d) Find the order of the following linear multistep method:

$$y_{n+2} - 2y_{n+1} + y_n = \frac{h}{2}(f_{n+2} - f_n). \tag{1}$$

e) Is the method (1) convergent?

f) Is the following matrix symmetric positive definite?

$$A = \begin{pmatrix} 3.2 & -1.2 & 0.8 \\ -1.2 & 4.6 & 1.2 \\ 0.8 & 1.2 & 3.6 \end{pmatrix}$$

g) Will the following linear iteration scheme converge?

$$\begin{aligned} x_1^{(k+1)} &= \frac{-x_1^{(k)} - x_2^{(k)} + x_3^{(k)} + 2}{4} \\ x_2^{(k+1)} &= \frac{2x_1^{(k)} + x_2^{(k)} - x_3^{(k)} - 1}{6} \\ x_3^{(k+1)} &= \frac{x_1^{(k)} - x_2^{(k)} + x_3^{(k)} + 4}{4} \end{aligned}$$

h) Is the following function a natural cubic spline?

$$S(x) = \begin{cases} x^3 - 1, & -1 \le x < 0 \\ 3x^3 - 1, & 0 \le x \le 1. \end{cases}$$

i) An ordinary differential equation $y' = f(t, y)$, $y(t_0) = y_0$ is solved by some Runge-Kutta method, using stepsizes $h = (t_{end} - t_0)/N$. The following table lists the global error $e_N = y_{tend} - y_N$ for different values of N.

N	20	40	80	160	320
e_N	$1.93 \cdot 10^{-6}$	$2.15 \cdot 10^{-7}$	$2.54 \cdot 10^{-8}$	$3.08 \cdot 10^{-9}$	$3.80 \cdot 10^{-10}$

From this experiment, what will you expect the order of the method to be?

Problem 2 (30 %)

a) Find the first two polynomials orthogonal to the inner product

$$\langle f, g \rangle = \int_0^\infty e^{-x} f(x) g(x) dx.$$

and prove that the third one is

$$\phi_2(x) = x^2 - 4x + 2.$$

Hint: $\int_0^\infty x^n e^{-x} dx = n!$

b) Use the result from **a)** to construct a quadrature formula of the form

$$\int_0^\infty e^{-x} f(x) dx \approx W_1 f(x_1) + W_2 f(x_2)$$

of optimal degree of precision (the Gauss quadrature).

c) Use the quadrature rule to find an approximation to the integral

$$\int_0^\infty e^{-x} \sin x \, dx.$$

Find an estimate (upper bound) for the error.

Problem 3 (20%)
Given the initial value problem

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}), \qquad \mathbf{y}'(t_0) = \mathbf{y}_0.$$

where $\mathbf{f} : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$. The trapezoidal rule for solving this ODE is given

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{2}\Big(\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) + \mathbf{f}(t_n, \mathbf{y}_n)\Big).$$

where $h = t_{n+1} - t_n$.

a) We will now apply the trapezoidal rule to the initial value problem

$$\begin{aligned} y' &= yz^2 + 2\sin(t), & y(0) &= 1, \\ z' &= 2y - z, & z(0) &= -2. \end{aligned}$$

Set up the set of nonlinear equations that has to be solved in each timestep.

Explain how to solve this system by Newton's method.
(You are supposed to write down the linear system that has to be solved for each iteration, but you do not have to explain how to solve it).

b) Supppose $\mathbf{f}$ satisfy the Lipschitz condition

$$\|\mathbf{f}(t, \mathbf{y}) - \mathbf{f}(t, \tilde{\mathbf{y}})\| \le L\|\mathbf{y} - \tilde{\mathbf{y}}\|, \qquad \text{for all } t \in \mathbb{R},\ \mathbf{y}, \tilde{\mathbf{y}} \in \mathbb{R}^m.$$

The local truncation error for the trapezoidal method

$$\mathbf{d}_{n+1} = \mathbf{y}(t_{n+1}) - \mathbf{y}(t_n) - \frac{h}{2}\Big(\mathbf{f}(t_{n+1}, \mathbf{y}(t_{n+1})) + \mathbf{f}(t_n, \mathbf{y}(t_n))\Big)$$

satisfies

$$\|\mathbf{d}_{n+1}\| \le \frac{1}{12}h^3 M, \qquad M = \max_{\xi \in \mathbb{R}} \|\mathbf{y}'''(\xi)\|.$$

Use this to show that the global error $\mathbf{e}_n = \mathbf{y}(t_n) - \mathbf{y}_n$ satisfies

$$\|\mathbf{e}_{n+1}\| \le \frac{1 + \frac{1}{2}hL}{1 - \frac{1}{2}hL}\|\mathbf{e}_n\| + \frac{\frac{1}{12}Mh^3}{1 - \frac{1}{2}hL} \qquad \text{for } hL < 2.$$

Finally, use this to prove the following estimate for the upper bound of the global error:

$$\|\mathbf{e}_n\| \le \frac{Mh^2}{12L}\left[\left(\frac{1 + \frac{1}{2}hL}{1 - \frac{1}{2}hL}\right)^n - 1\right],$$

assuming $\mathbf{e}_0 = 0$.