Problem 1:

a)

$$p_2(x) = 3\frac{(x-3)(x-4)}{(1-3)(1-4)} + \frac{(x-1)(x-4)}{(3-1)(3-4)} + 8\frac{(x-1)(x-3)}{(4-1)(4-3)}.$$

b) We have

$$x_{k+1} = g(x_k), \qquad g(x) = \frac{e^x - 1}{2}$$

We also find

$$g'(x) = \frac{1}{2}e^x, \qquad g'(x^*) = 1.76$$

So $|g'(x^*)| > 1$, x^* is an unstable fixed point, and the iterations will not converge.

c) $Q(f)$ has degree of accuracy p if

$$Q(x^k) = \int_0^1 x^k dx \quad \text{for } k=0,\cdots,p, \text{ but not for } k=p+1.$$

$$x^0: \int_0^1 dx = 1, \quad Q(1) = \frac{1}{4}(3+1) = 1$$

$$x: \int_0^1 x\,dx = \frac{1}{2}, \quad Q(x) = \frac{1}{4}\left(3\cdot\frac{1}{3} + 1\cdot 1\right) = \frac{1}{2}$$

$$x^2: \int_0^1 x^2 dx = \frac{1}{3}, \quad Q(x^2) = \frac{1}{4}\left(3\cdot\frac{1}{9} + 1\cdot 1\right) = \frac{1}{3}$$

$$x^3: \int_0^1 x^3 dx = \frac{1}{4}, \quad Q(x^3) = \frac{1}{4}\left(3\cdot\frac{1}{27} + 1\cdot 1\right) = \frac{5}{18}$$

So the quadrature has precision 2

d) A LMM is of order p if

$$h\tau_{n+k} = \sum_{\ell=0}^{k} (\alpha_\ell y(t_{n+\ell}) - h\beta_\ell y'(t_{n+\ell})) = C_{p+1} h^{p+1} + O(h^{p+2})$$

In our case, this is

$$\begin{aligned} h\tau_{n+2} &= y(t_n+2h) - 2y(t_n+h) + y(t_n) - \frac{h}{2}(y'(t_n+2h) - y'(t_n)) \\ &= y + 2hy' + \frac{1}{2}(2h)^2 y'' + \frac{1}{6}(2h)^3 y''' + \frac{1}{24}(2h)^4 y^{(iv)} + \cdots \\ &\quad - 2(y + hy' + \frac{1}{2}h^2 y'' + \frac{1}{6}h^3 y''' + \frac{1}{24}h^4 y^{(iv)} + \cdots) \\ &\quad + y \\ &\quad - \frac{h}{2}((2h)y'' + \frac{1}{2}(2h)^2 y''' + \frac{1}{6}(2h)^3 y^{(iv)} + \cdots) \\ &= -\frac{1}{12}h^4 y^{(iv)} + \cdots \end{aligned}$$

The method is of order 3.

e) For the method to be convergent, it has to be consistent and zero-stable. Since $p=3$, it is consistent.
For zero-stability, we have to find the roots of the characteristic polynomial

$$\rho(r) = \sum_{\ell=0}^{k} \alpha_\ell r^\ell = r^2 - 2r + 1 = (r-1)^2$$

So, $\rho(r)$ has a double root, $r=1$ on the unit circle $\Rightarrow$ the method is not zero-stable and thus not convergent.

f) Yes, it is SPD, since it is symmetric and diagonal dominant

$$a_{ii} > \sum_{\substack{j=1 \\ j\neq i}}^{3} |a_{ij}| \quad , \quad i = 1,2,3$$

with positive diagonal elements.
More specific:

$3.2 > 2.0, \quad 4.6 > 2.4, \quad 3.6 > 2.0.$

g) The iteration scheme can be written as

$$\vec{x}^{(k+1)} = G x^{(k)} + b$$

with
$$G = \begin{pmatrix} -1/4 & -1/4 & 1/4 \\ 1/3 & 1/6 & -1/6 \\ 1/4 & -1/4 & 1/4 \end{pmatrix}, \quad b = \begin{pmatrix} 1/2 \\ -1/6 \\ 1 \end{pmatrix}$$

The iterations converges if $\|G\| < 1$ in some norm. Let us start with $\|\cdot\|_\infty$, and

$$\|G\|_\infty = \max\left(\frac{3}{4}, \frac{2}{3}, \frac{3}{4}\right) = 0.75 < 1$$

So yes, this scheme converges.

h) It is a cubic spline if $S \in C^2[-1,1]$

$S(0^-) = S(0^+)$: $0^3 - 1 = 3\cdot 0^3 - 1$ OK

$S'(0^-) = S'(0^+)$: $3\cdot 0^2 = 9\cdot 0^2$ OK

$S''(0^-) = S''(0^+)$: $6\cdot 0 = 18\cdot 0$ OK

So yes, it is a cubic spline.

It is a natural cubic spline if

$$S''(-1) = S''(1) = 0$$

This is not the case, so it is not a natural cubic spline

i) If the method is of order p, we expect that
$$e_N \approx C\cdot h^p = C\cdot \frac{t_{end} - t_0}{N}$$

or
$$\frac{|e_{N/2}|}{|e_N|} \approx 2^p.$$

From the table, we observe

$$\frac{e_{20}}{e_{40}} = 8.98, \quad \frac{e_{40}}{e_{80}} = 8.46, \quad \frac{e_{80}}{e_{160}} = 8.25, \quad \frac{e_{160}}{e_{320}} = 8.11$$

So we conclude that the method is of order 3.

Problem 2:

a) See the note, Theorem 4.1

$\varphi_0(x) = 1, \quad \langle x\varphi_0, \varphi_0 \rangle = 1, \quad \langle \varphi_0, \varphi_0 \rangle = 1 \quad \Rightarrow \quad B_1 = 1$

$\varphi_1(x) = x - 1, \quad \langle x\varphi_1, \varphi_1 \rangle = 3, \quad \langle \varphi_1, \varphi_1 \rangle = 1 \quad \Rightarrow \quad B_2 = 3, \quad C_2 = 1$

$\varphi_2(x) = x(x-1) - 3(x-1) - 1 = x^2 - 4x + 2$

b) The roots of $\varphi_2(x)$ are $r_1 = 2 - \sqrt{2}$, $r_2 = 2 + \sqrt{2}$

Let $p_2(x) = f(r_1)\dfrac{x - 2 - \sqrt{2}}{-2\sqrt{2}} + f(r_2)\dfrac{x - 2 + \sqrt{2}}{2\sqrt{2}}$

The Gauss quadrature formula is

$$Q(f) = \int_0^\infty e^{-x} p_2(x)\,dx$$

$$= f(r_1) \cdot \int_0^\infty e^{-x} \frac{x - 2 - \sqrt{2}}{-2\sqrt{2}}\,dx + f(r_2)\int_0^\infty e^{-x} \frac{x - 2 + \sqrt{2}}{2\sqrt{2}}\,dx$$

$$= \frac{2 + \sqrt{2}}{4} f(2 - \sqrt{2}) + \frac{2 - \sqrt{2}}{4} f(2 + \sqrt{2}).$$

c) $$\int_0^\infty e^{-x}\sin(x)\,dx \approx \frac{2+\sqrt{2}}{4}\sin(2 - \sqrt{2}) + \frac{2 - \sqrt{2}}{4}\sin(2 + \sqrt{2}) = 0.4325$$

For the error bound, we have (Thm. 4.7 in the note)

$$E(f) = \frac{f^{4}(\xi)}{4!}\int_0^\infty e^{-x}(x - 2 + \sqrt{2})^2(x - 2 - \sqrt{2})^2\,dx$$

$$= \frac{f^{(4)}(\xi)}{3!} \quad \text{for some } \xi \in (0, \infty)$$

And, since $f^{(4)}(x) = \sin(x)$, we get the following estimate:

$$|E(f)| \leq \frac{1}{6}.$$

(Only for information. The value of the integral is 0.5, so the error is 0.068, well beyond the theoretical bound.)

Problem 3

a) y_n, z_n are known.

$$y_{n+1} = y_n + \frac{h}{2}\left(y_{n+1}\cdot z_{n+1}^2 - 2\sin(t_{n+1}) + y_n\cdot z_n^2 - 2\sin(t_n)\right).$$

$$z_{n+1} = z_n + \frac{h}{2}\left(2y_{n+1} - z_{n+1} + 2y_n - z_n\right)$$

For each step, this system is solved wrt. y_{n+1}, z_{n+1}, by Newtons method:

$$J_k \cdot \begin{pmatrix} \Delta y_{n+1}^{(k)} \\ \Delta z_{n+1}^{(k)} \end{pmatrix} = \begin{pmatrix} -y_{n+1}^{(k)} + \frac{h}{2}\left(y_{n+1}^{(k)}\cdot z_{n+1}^{(k)\,2} + y_n z_n^2 + 2\sin(t_{n+1}) + 2\sin(t_n)\right) \\ -z_{n+1}^{(k)} + \frac{h}{2}\left(2y_{n+1}^{(k)} - z_{n+1}^{(k)} + 2y_n - z_n\right) \end{pmatrix}$$

where

$$J_k = \begin{pmatrix} 1 - \frac{h}{2} z_{n+1}^{(k)} & h y_{n+1}^{(k)} z_{n+1}^{(k)} \\ h y_{n+1}^{(k)} & 1 - \frac{h}{2} \end{pmatrix}$$

and

$$y_{n+1}^{(k+1)} = y_{n+1}^{(k)} + \Delta y_{n+1}^{(k)}$$

$$z_{n+1}^{(k+1)} = z_{n+1}^{(k)} + \Delta z_{n+1}^{(k)}.$$

As starting values, use $y_{n+1}^{(0)} = y_n$, $z_{n+1}^{(0)} = z_n$.

Stop the iterations if

$$\max\left(\Delta y_{n+1}^{(k)}, \Delta z_{n+1}^{(k)}\right) \le Tol \quad \text{(convergence)}$$

or if a maximum number of iterations are done (divergence).

Other stopping criterias can also be accepted.

Problem 3:

b) We have:

$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_{n+1}, y_{n+1}) + f(t_n, y_n)\right)$$

$$y(t_{n+1}) = y(t_n) + \frac{h}{2}\left(f(t_{n+1}, y(t_{n+1})) + f(t_n, y(t_n))\right) + d_{n+1}$$

Subtracting the first from the second gives

$$e_{n+1} = e_n + \frac{h}{2}\left(f(t_{n+1}, y(t_{n+1})) - f(t_{n+1}, y_{n+1})\right.$$
$$\left. + f(t_n, y(t_n)) - f(t_n, y_n)\right) + d_{n+1}$$

By taking the norm on both sides, using the triangle inequality and using the Lipschitz conditions, this becomes

$$\|e_{n+1}\| \leq \|e_n\| + \frac{hL}{2}\left(\|e_{n+1}\| + \|e_n\|\right) + \|d_{n+1}\|.$$

or

$$\left(1 - \frac{hL}{2}\right)\|e_{n+1}\| \leq \left(1 + \frac{hL}{2}\right)\|e_n\| + \frac{h^3}{12} \cdot M$$

As long as $1 - \frac{hL}{2} > 0$, dividing by this expression on both sides gives the expected result.

For simplicity, let

$$K = \frac{1 + \frac{1}{2}hL}{1 - \frac{1}{2}hL}, \qquad T = \frac{1}{12}\frac{M}{1 - \frac{1}{2}hL}.$$

We get: $(e_0 = 0)$

$$\|e_1\| \leq T, \quad \|e_2\| \leq K \cdot \|e_1\| + T \leq (K+1) \cdot T$$

Continue like this, and we get

$$\|e_n\| \leq \left(\sum_{\ell=0}^{n-1} K^{\ell}\right) \cdot T = \frac{K^n - 1}{K - 1} \cdot T$$

and since

$$\frac{1}{K-1} = \frac{1 - \frac{hL}{2}}{hL} < \frac{1}{hL}$$

we get

$$\|e_n\| \leq \frac{Mh^2}{12L}\left[\left(\frac{1 + \frac{hL}{2}}{1 - \frac{hL}{2}}\right)^n - 1\right].$$