Problem 1 (Counts 50%)

a) Use the error propagation law for $y = \varphi(x)$

$$\delta y = \frac{\partial \varphi}{\partial \varphi} \frac{x}{\varphi} \delta x$$

which for $\varphi(x) = \ln(x)$ becomes

$$\delta y = \frac{1}{\ln(x)} \delta x.$$

b)

$$\begin{aligned} k_1 &= f(t_n, y_n) \\ k_2 &= f(t_n + \tfrac{2h}{3}, y_n + \tfrac{2h}{3} k_1) \\ y_{n+1} &= y_n + \tfrac{h}{4}(k_1 + 3k_2) \end{aligned}$$

c) We know that an explicit method of s-stages can be of maximum order s. So in this case, we only have to check that the method is of order 2:

$$\begin{aligned} &\text{Order 1 :} \qquad & b_1 + b_2 &= \frac{1}{4} + \frac{3}{4} = 1 \\ &\text{Order 2 :} \qquad & b_1 c_1 + b_2 c_2 &= \frac{1}{4} \cdot 0 + \frac{3}{4} \cdot \frac{3}{2} = \frac{1}{2} \end{aligned}$$

The order of the method is 2.

d) The order of the tree is the number of nodes, thus $\rho(\tau) = 6$. The order condition becomes:

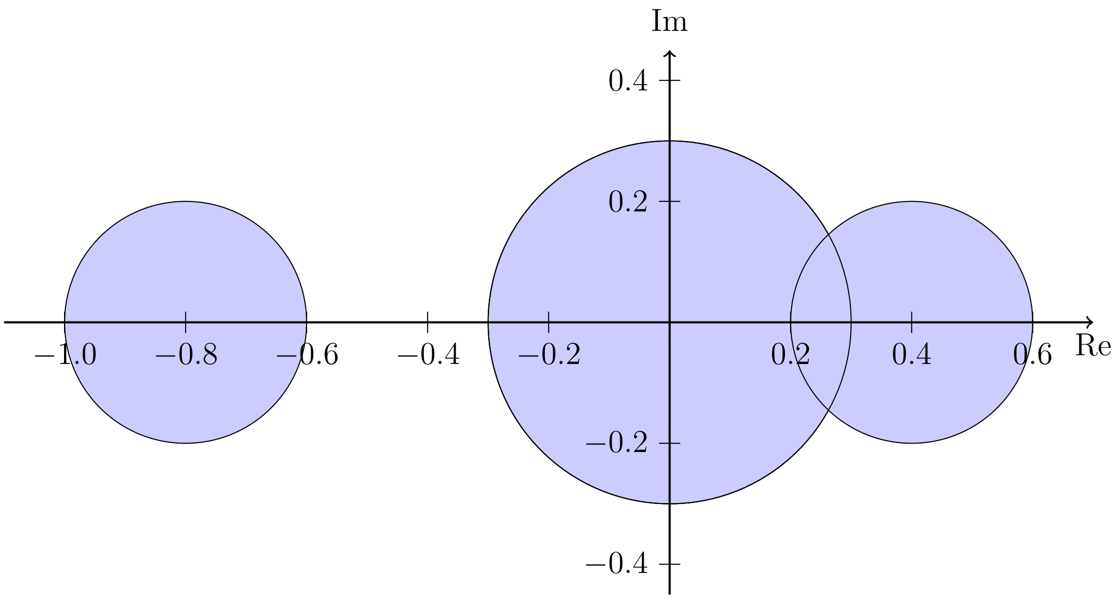
$$\sum_{i,j,k=1}^{s} b_i a_{ij} c_j^2 a_{ik} c_k = \frac{1}{36}.$$

e) The Gauss-Seidel scheme is given by

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{4}(1 + 2x_2^{(k)}) \\ x_2^{(k+1)} &= \frac{1}{3}(4 - x_1^{(k+1)} + x_3^{(k)}), \qquad k = 0, 1, 2, 3 \ldots \\ x_3^{(k+1)} &= -\frac{1}{2}(-2 - x_2^{(k+1)}) \end{aligned}$$

The coefficient matrix is strict diagonal dominant,($4 > 2$, $3 > 2$ and $2 > 1$) so yes, the iterations will converge for all choices of initial values.

f) The Gershgorin disks are:



from which we can conclude that $\rho(A) = \max_{i=1,2,3} |\lambda_i| \leq 1$.

g) We know that orthogonal matrices preserves the 2-norm, that is $\|Qx\|_2 = \|x\|_2$. So $\alpha = \|x\|_2 = 5$. Using Householder reflections, we have

$$\tilde{\boldsymbol{v}} = \boldsymbol{x} - \alpha \boldsymbol{e}_1 = \begin{bmatrix} -3 \\ 4 \end{bmatrix} - \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}, \qquad \boldsymbol{v} = \frac{\tilde{\boldsymbol{v}}}{\|\tilde{\boldsymbol{v}}\|} = \frac{1}{4\sqrt{5}} \begin{bmatrix} -8 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Finally

$$Q = I - 2\boldsymbol{v}\boldsymbol{v}^T = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

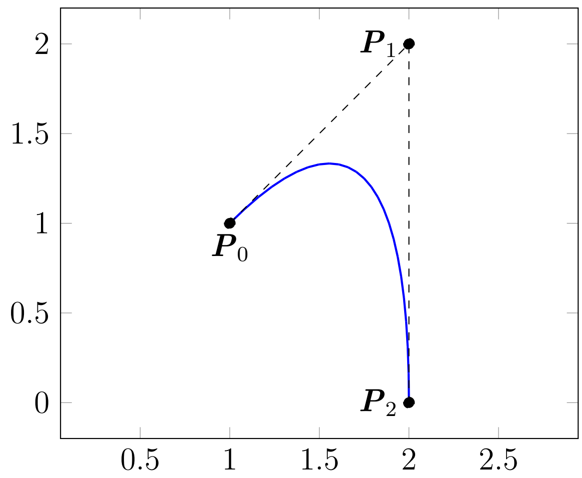
h) This is a cubic spline if S is two times continuous differentiable. We only have to check this for $x = 2$. Let $S_1 = x^2/2$ and $S_2 = x^3/2 - 5x^2/2 + 5x - 2$.

$$\begin{aligned} S_1(2) &= 2 & \qquad S_2(2) &= 2 \\ S_1'(2) &= 2 & \qquad S_2'(2) &= 1 \end{aligned}$$

so this is not a cubic spline.

i) The parametric quadratic Bézier-curve is

$$\boldsymbol{B}(t) = \boldsymbol{P}_0\, b_{2,0}(t) + \boldsymbol{P}_1\, b_{2,1}(t) + \boldsymbol{P}_2\, b_{2,2}(t) = \begin{bmatrix} -t^2 + 2t + 1 \\ -3t^2 + 2t + 1 \end{bmatrix}.$$



Problem 2 (Counts 20%)

a)

$$F(0.1) = \frac{1}{0.1^3}\left(-\frac{1}{2}\mathrm{e}^{-0.2} + \mathrm{e}^{-0.1} - \mathrm{e}^{0.1} + \frac{1}{2}\mathrm{e}^{0.2}\right) = 1.0025025$$

b) Expand $F(h)$ by Taylor expansions:

$$\begin{aligned} F(h) &= \frac{1}{h^3}\left(\frac{1}{2}\Big(f(x_0+2h) - f(x_0-2h)\Big) - \Big(f(x_0+h) - f(x_0-h)\Big)\right) \\ &= \frac{1}{h^3}\left(\sum_{p=0}^{\infty}\frac{1}{2}(1-(-1)^p)\frac{(2h)^p}{p!}f^{(p)}(x_0) - \sum_{p=0}^{\infty}(1-(-1)^p)\frac{h^p}{p!}f^{(p)}(x_0)\right) \\ &= \sum_{q=1}^{\infty}\frac{2}{(2q+1)!}(2^{2q}-1)f^{(2q+1)}(x_0)h^{2(q-1)} \\ &= f^{(3)}(x_0) + \frac{1}{4}f^{(5)}(x_0)h^2 + C_4h^4 + \cdots . \end{aligned}$$

so $C_4 = f^{(4)}(x_0)/4$.

c) Use extrapolation: Since

$$F(h) = f^{(3)}(x_0) + C_2h^2 + C_4h^4 + C_6h^6 + \cdots$$

we can show that

$$G(h) = \frac{1}{3}\Big(4F(\tfrac{h}{2}) - F(h)\Big) = f^{(3)}(x_0) + D_4h^4 + D_6h^6 + \cdots$$

and

$$H(h) = \frac{1}{15}\Big(16F(\tfrac{h}{2}) - F(h)\Big) = f^{(3)}(x_0) + K_6 h^6 + \cdots$$

where D_q and K_q are some constants independent of h. From this we can find

$$\begin{aligned}
&F(h) = 26.69065 \\
&F(\tfrac{h}{2}) = 28.76826 \qquad && G(h) = 29.46080 \\
&F(\tfrac{h}{4}) = 29.30727 \qquad && G(\tfrac{h}{2}) = 29.48694 \qquad && H(h) = 29.48868
\end{aligned}$$

where $h = 0.4$. So the best approximation we can get from the available data is 29.48868. (For comparision, the correct answer is 29.48872149.)

Problem 3

a) From (1) in the appendix infer that

$$q(t) = q(t_{n+1} + sh) = y_{n+1} + \sum_{j=1}^{k} (-1)^j \binom{-s}{j} \nabla^j y_{n+1},$$

and so an application of the chain rule, using that

$$t(s) = t_{n+1} + sh \qquad \text{and} \qquad s(t) = \frac{t - t_{n+1}}{h},$$

results in

$$q'(t_{n+1}) = \frac{\mathrm{d}q(t(s))}{\mathrm{d}s}\bigg|_{s=0} \frac{\mathrm{d}s}{\mathrm{d}t}\bigg|_{t=t_{n+1}} = \sum_{j=1}^{k} (-1)^j \left(\frac{\mathrm{d}}{\mathrm{d}s} \binom{-s}{j} \bigg|_{s=0} \right) \nabla^j y_{n+1} \cdot \frac{1}{h}.$$

Thus we obtain the desired form since

$$(-1)^j \frac{\mathrm{d}}{\mathrm{d}s} \binom{-s}{j} \bigg|_{s=0} = \frac{1}{j}.$$

For full score, this should be proved!

b) We have:

$$\sum_{j=1}^{3} \tfrac{1}{j} \nabla^j y_{n+1} = (y_{n+1} - y_n) + \tfrac{1}{2}(y_{n+1} - 2y_n + y_{n-1})$$

so the 2-step BDF -method is given by

$$\tfrac{3}{2} y_{n+1} - 2y_n + \tfrac{1}{2} y_{n-1} = h f_{n+1}.$$

The characteristic polynomial is

$$\rho(r) = \tfrac{3}{2}r^2 - 2r + \tfrac{1}{2} = \tfrac{3}{2}(r-1)(r-\tfrac{1}{3})$$

So $r_i \leq 1$ for $i = 1, 2$, and the one of length 1 is single. In conclusion, the method is zero-stable.

The local discretization error is defined by

$$h\tau_{n+1} = \tfrac{3}{2}y(t_{n-1}+2h) - 2y(t_{n-1}+h) + \tfrac{3}{2}y(t_{n-1}) - hy'(t_{n-1}+2h).$$

By taylor-expansions around t_{n-1} we get

$$\begin{aligned} h\tau_{n+1} &= \tfrac{3}{2}\Big(y + 2hy' + \tfrac{2^2}{2}h^2y'' + \tfrac{2^3}{6}h^3 + \cdots\Big) \\ &- 2\Big(y + hy' + \tfrac{1}{2}h^2y'' + \tfrac{1}{6}h^3y''' + \cdots\Big) \\ &+ \tfrac{1}{2}y \\ &- hy' - 2h^2y'' - \tfrac{2^2}{2}h^3y''' - \tfrac{2^3}{6}h^4y^{(4)} - \cdots \\ &= -\tfrac{1}{3}h^3y^{(3)} + \cdots. \end{aligned}$$

The method is of order 2, and the error constant $C_3 = -1/3$.

Problem 4 The second equation can be rewritten as $x_2 = (\sin x_1 + \cos x_2)/4$. The first equation can be solved with respect to x_1, and since we only are interested in the positive solution, we try

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{5}}x_2 \\ x_2 &= \frac{1}{4}\Big(\sin x_1 + \cos x_2\Big). \end{aligned}$$

or

$$\boldsymbol{G}(\boldsymbol{x}) = (g_1(x_1,x_2), g_2(x_1,x_2)] = \begin{bmatrix} \frac{1}{\sqrt{5}}x_2 \\ \frac{1}{4}\Big(\sin x_1 + \cos x_2\Big). \end{bmatrix}$$

Let us now suggest a domain D. First, we assume both solutions to be positive, so $a = c = 0$. From the second equation we get that $x_2 \leq 1/2$, which again mean that $x_1 \leq 1/(2\sqrt{5})$. So

$$D = \{\boldsymbol{x} \in \mathbb{R}^2 \;:\; 0 \leq x_1 \leq \frac{1}{2\sqrt{5}}, 0 \leq x_2 \leq \frac{1}{2}\}$$

seems like a reasonable choice. If we can show that

$$\boldsymbol{G}(D) \in D \qquad \text{and} \qquad \|\,\boldsymbol{G}(\boldsymbol{y}) - \boldsymbol{G}(\boldsymbol{v})\| \leq L\|\boldsymbol{y} - \boldsymbol{v}\| \quad \text{with} \quad L < 1 \quad \forall \boldsymbol{x}, \boldsymbol{y} \in D.$$

then Banach's fixed-point theorem guarantees convergence towards a unique fixed point $\boldsymbol{x} \in D$.

From the construction of D we know that the first condition is satisfied as long as $0 \leq g_2(x_1, x_2) \leq 1$, which is true since $0 \leq \sin x_1 < 1$ and $0 < \cos x_2 \leq 1$ for all $(x_1, x_2) \in D$ (sharper bounds can be found, but are not required.) Using the mean value theorem, we know that the second condition is satified in the max-norm if

$$L = \max_{i=1,2} \sum_{j=1}^{2} \left| \frac{\partial g_i}{\partial x_j}(\boldsymbol{x}) \right| < 1, \qquad \forall \boldsymbol{x} \in D.$$

The Jacobian of $\boldsymbol{G}$

$$J_{\boldsymbol{G}} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\sqrt{5}} \\ \frac{1}{4}\cos x_1 & -\frac{1}{4}\sin x_2 \end{bmatrix}$$

and

$$L = \max\left\{ \frac{1}{\sqrt{5}}, \frac{1}{4}\left(1 + \sin\frac{1}{2}\right) \right\} = \max\{0.4472, 0.3699\} = 0.4472 < 1.$$