NTNU
Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4215 Numerical Mathematics**

Academic contact during examination: Anne Kværnø
Phone: 92663824

Examination date: December 9, 2016
Examination time (from–to): 09:00–13:00
Permitted examination support material: C:

- Rottmann: *Matematisk formelsamling*
- Approved calculator
- One yellow, stamped A5 sheet with own handwritten formulas and notes (on both sides).

Language: English
Number of pages: 4
Number of pages enclosed: 1

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
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skal ha flervalgskjema ☐	

Date Signature

NB! There is an appendix at the end of the set!

Problem 1 (Counts 50%)

In this problem, you are given small exercises from different parts of the curriculum. *Note that yes/no answers without justification give no credit.*

a) Let the function $y = \ln x$ be given and assume that the relative error of x is δx. What is the corresponding relative error δy of y?

b) A Runge–Kutta method has a Butcher-tableau

$$\begin{array}{c|cc} 0 & & \\ 2/3 & 2/3 & \\ \hline 0 & 1/4 & 3/4 \end{array}$$

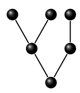
State the corresponding algorithm when this method is applied to the intial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0,$$

using stepsize h.

c) Find the order of the method in **b)**.

d) State the order of the rooted tree

$\tau =$ 

and find the order condition $\varphi(\tau) = 1/\gamma(\tau)$ corresponding to τ for a general s–stage Runge–Kutta method.

e) Set up the Gauss–Seidel iteration scheme for the linear system

$$\begin{aligned} 4x_1 - 2x_2 \qquad &= 1 \\ x_1 + 3x_2 - x_3 &= 4 \\ x_2 - 2x_3 &= -2 \end{aligned}$$

Will the scheme converge for any starting value $\boldsymbol{x}^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) \in \mathbb{R}^3$?

f) Sketch the row-wise Gershgorin disks for the matrix

$$A = \begin{bmatrix} 0.0 & 0.2 & -0.1 \\ -0.2 & -0.8 & 0.0 \\ -0.1 & -0.1 & 0.4 \end{bmatrix}$$

and give an upper bound for the spectral radius $\rho(A)$.

g) Given the vector $\boldsymbol{x} = [-3\ \ 4]^\top$, find an orthogonal matrix Q via Householder reflection such that

$$Q\boldsymbol{x} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$$

for some constant $\alpha > 0$. What is α in this case?

h) Is the function

$$S(x) = \begin{cases} \frac{1}{2}x^2, & x \in [0, 2), \\ \frac{1}{2}x^3 - \frac{5}{2}x^2 + 5x - 2, & x \in [2, 4] \end{cases}$$

a cubic spline?

i) Write up and sketch the parametric quadratic Bézier-curve given by the control points

$$\boldsymbol{P}_0 = (1, 1), \qquad \boldsymbol{P}_1 = (2, 2) \qquad \text{and} \qquad \boldsymbol{P}_2 = (2, 0).$$

Problem 2 (Counts 20%)

Let h be some stepsize and consider the formula

$$F(h) = \frac{1}{h^3}\left[-\tfrac{1}{2}f(x_0 - 2h) + f(x_0 - h) - f(x_0 + h) + \tfrac{1}{2}f(x_0 + 2h)\right] \tag{1}$$

used to approximate the third derivative $f^{(3)}$ at a point x_0 of a sufficiently smooth function f.

a) Apply the formula to $f(x) = \mathrm{e}^x$ at $x_0 = 0$ with $h = 0.1$.

b) Show that $F(h)$ can be expanded as

$$F(h) = f^{(3)}(x_0) + C_2h^2 + C_4h^4 + \cdots$$

with only even powers of h, and find an expression for C_2.

c) A student applied formula (1) to a smooth function f at some point x_0 for different values of h, which gave the following results:

h	0.4	0.2	0.1
$F(h)$	26.69065	28.76826	29.30727

Use this to make a better (optimal) approximation to $f^{(3)}(x_0)$.

Problem 3 (Counts 20%)

The *backward differentiation formulas* (BDFs) is a family of implicit linear k-step methods for the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0,$$

where we assume that the starting values $y_1, \dots, y_{k-1}$ (in addition to y_0) are given with sufficient accuracy. To construct them, we first use (2) from the Appendix to find the kth degree polynomial q interpolating all the previously computed points *plus* the new one, that is, the points

$$(t_{n-k+1}, y_{n-k+1}),\ \dots,\ (t_n, y_n),\ (t_{n+1}, y_{n+1}).$$

Then we replace y by q in the differential equation and evaluate the corresponding equation at t_{n+1}, which yields the BDF

$$q'(t_{n+1}) = f(t_{n+1}, y_{n+1}).$$

a) Use this to prove that a k-step BDF can be written as

$$\sum_{j=1}^{k} \frac{1}{j} \nabla^j y_{n+1} = h f_{n+1},$$

where $f_{n+1} := f(t_{n+1}, y_{n+1})$.

b) Write down the method when $k = 2$, and check if it is zero-stable. Find the order of the method and the corresponding error constant.

Problem 4 (Counts 10%)

The nonlinear system

$$5x_1^2 - x_2^2 = 0$$

$$x_2 - \frac{1}{4}(\sin x_1 + \cos x_2) = 0$$

has a solution $\boldsymbol{x} = (x_1, x_2)$ close to $(0.1, 0.25)$.

Find a vector-function $\boldsymbol{G}$ and a rectangular set

$$D = \{\boldsymbol{x} \in \mathbb{R}^2 : a \leq x_1 \leq b \text{ and } c \leq x_2 \leq d\},$$

for some suitable values $a, b, c, d \in \mathbb{R}$, such that the fixed point iteration

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{G}(\boldsymbol{x}^k)$$

converges to $\boldsymbol{x} \in D$ for all starting values $\boldsymbol{x}^{(0)} \in D$.

Appendix

- A general s-stage Runge–Kutta method has a Butcher-tableaux

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & & \vdots \\ c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\ \hline & b_1 & b_2 & \cdots & b_s \end{array}$$

- The Bernstein polynomials are given by

$$b_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}.$$

- Let $\{t_i\}_{i=0}^m$ be a set of equidistant nodes, so that $t_i = t_0 + ih$ for $i = 1, 2, \ldots, m$, where h is the distance between the nodes. Given a function f defined on an interval containing the nodes, with $f_i := f(t_i)$ for $i = 0, 1, \ldots, m$, we can introduce *backward differences* recursively by

  $$\nabla^0 f_i = f_i, \qquad \nabla f_i = f_i - f_{i-1}, \qquad \nabla^j f_i = \nabla^{j-1} f_i - \nabla^{j-1} f_{i-1}, \qquad j = 1, 2, \ldots$$

  Now let $t = t_0 + sh$, where $s \in \mathbb{R}$, be any point. The mth degree polynomial p interpolating f in the nodes $\{t_i\}$ can be written as

  $$p(t) = p(t_m + sh) = f_m + \sum_{j=1}^{m} (-1)^j \binom{-s}{j} \nabla^j f_m, \tag{2}$$

  and (2) is known as *Newton's backward difference formula*. Here

  $$\binom{s}{j} := \frac{s(s-1)\cdots(s-j+1)}{j!}$$

  is a generalized binomial coefficient.