More on splines

Given the knots $t_0 < t_1 < \cdots < t_n$.
Spline of degree k:

1) On each interval $[t_i; t_{i+1})$, $S \in \mathbb{P}_k$
2) $S \in C^{k-1}[t_0, t_n]$

Let S_n^k be the space of all splines of degree k, and let us find a basis for it.

Dimension:

$n(k+1)$ variables
$(n-1)\cdot k$ restrictions

So a qualified guess is that $\dim S_n^k = n+k$.

Basis:

Let $$(x-t_i)_+^k = \begin{cases} (x-t_i)^k & \text{if } x \geq t_i \\ 0 & x < t_i \end{cases}$$

Notice that $(x-t_i)_+^k \in C^{(k-1)}[t_0, t_n]$

since $\frac{d^j}{dx^j}(x-t_i)^k \Big|_{x=t_i} = 0$, $j = 0, 1, \cdots, k-1$

So, we can write:

$$S(x) = \sum_{i=0}^{k} a_i x^i \quad \text{on} \quad [t_0, t_1)$$

$$\sum_{i=0}^{k} a_i x^i + b_1 (x-t_1)_+^k \quad \text{on} \quad [t_1, t_2]$$

$$\vdots$$

$$S(x) = \sum_{i=0}^{n} a_i x^i + \sum_{i=1}^{n-1} b_i (x-t_i)_+^k$$

and $S_n^k = \operatorname{span}\{1, x, \cdots, x^n, (x-t_1)_+^k, \cdots (x-t_{n-1})_+^k\}$.

Unfortunately, this is badly conditioned, and should be avoided in numerical work.

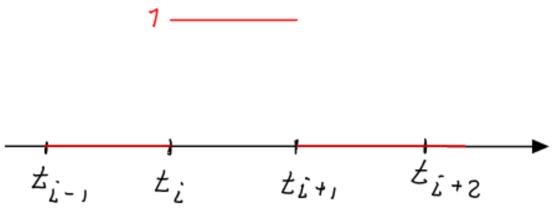
B-splines

Extend the set of knots:

$$\cdots < t_{-2} < t_{-1} < t_0 < t_1 < \cdots < t_n < t_{n+1} < \cdots$$

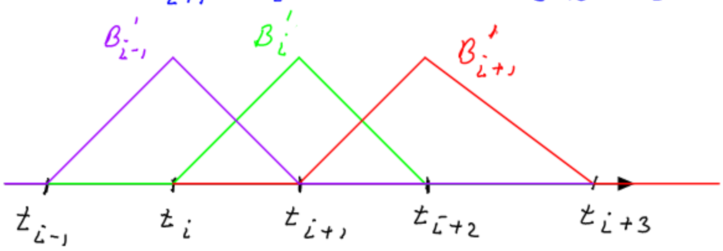
$k=0$:

$$B_i^0(x) = \begin{cases} 1 & \text{if } t_i \leqslant x < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$



$k=1$:

$$B_i^1(x) = \frac{x - t_i}{t_{i+1} - t_i} B_i^0(x) + \frac{t_{i+2} - x}{t_{i+2} - t_{i+1}} B_{i+1}^0(x)$$



Properties:

Support: supp (f) $= \{x \in \mathbb{R},\ f \neq 0\}$

1) $\operatorname{supp}(B_i^1) = (t_i, t_{i+2})$
2) $B_i^1(x) \geqslant 0$
3) $B_i^1 \in C^0(\mathbb{R})$ (and C^∞ everywhere except t_i, t_{i+1}, t_{i+2})
4) $\sum_{i=-\infty}^{\infty} B_i^1(x) = 1 \quad \forall x \in \mathbb{R}$

Proof of 4): Given some $x \in \mathbb{R}$, $\exists j$ s.t. $t_j \leqslant x < t_{j+1}$. Only $B_{j-1}^1(x)$ and $B_j^1(x)$ are possibly nonzero.

$$\sum_{i=-\infty}^{\infty} B_i^1(x) = B_j^1(x) + B_{j-1}^1(x)$$
$$= \frac{t_{j+1} - x}{t_{j+1} - t_j} + \frac{x - t_j}{t_{j+1} - t_j} = 1$$

General k:

$$B_i^k(x) = \frac{x - t_i}{t_{i+k-1} - t_i} B_i^{k-1}(x) + \frac{t_{i+k} - x}{t_{i+k} - t_{i+1}} B_{i+1}^{k-1}(x)$$

Properties: $(k \geq 2)$

1) $\operatorname{supp}(B_i^k) = (t_i, t_{i+k})$
2) $B_i^k(x) > 0$ on (t_i, t_{i+k})
3) $B_i^k \in C^{k-1}(\mathbb{R})$
4) $\sum_{i=-\infty}^{\infty} B_i(x) = 1$ (Partition of unity)

Numerical evaluation: (Carl de Boor)

$$f(x) = \sum_{i=-\infty}^{\infty} c_i B_i^k(x) \quad \text{given.}$$

Let $V_i^k(x) = \frac{x - t_i}{t_{i+k-1} - t_i}$, $1 - V_{i+1}^k(x) = \frac{t_{i+k} - x}{t_{i+k} - t_{i+1}}$

dvs.
$$B_i^k = V_i^k B_i^{k-1} + (1 - V_{i+1}^k) B_{i+1}^{k-1}$$

Then
$$\sum_{i=-\infty}^{\infty} c_i B_i^k = \sum_{i=-\infty}^{\infty} c_i^k B_i^k$$

$$= \sum_{i=-\infty}^{\infty} c_i^k (V_i^k B_i^{k-1} + (1 - V_{i+1}^k) B_{i+1}^{k-1})$$

$$= \sum_{i=-\infty}^{\infty} c_i^k V_i^k B_i^{k-1} + \sum_{i=-\infty}^{\infty} c_{i-1}^k (1 - V_i^k) B_i^{k-1}$$

$$= \sum_{i=-\infty}^{\infty} c_i^{k-1} B_i^{k-1},$$

$$\vdots$$

$$= \sum_{i=-\infty}^{\infty} c_i^0 B_i^0$$

where

(*)
$$\boxed{\begin{aligned} c_i^{k-1} &= c_i^k V_i^k + c_{i-1}^k (1 - V_i^k) \\ &= \frac{1}{t_{i+k} - t_i} \left[(x - t_i) c_i^k + (t_{i+k} - x) c_{i-1}^k \right] \end{aligned}}$$

Given some $x \in [t_j, t_{j+1}]$. To calculate

$$s(x) = \sum_{i=-\infty}^{\infty} c_i B_i^k(x) = \sum_{i=j-k}^{j} c_i B_i^k(x)$$

make the table

$$\begin{tikzcd}[row sep=small]
c_j = c_j^k \arrow[dr] & & \\
c_{j-1} = c_{j-1}^k \arrow[ur] \arrow[dr] & c_j^{k-1} \arrow[dr] & \\
c_{j-2} = c_{j-2}^k \arrow[ur] & c_{j-1}^{k-1} \arrow[ur] \arrow[r, dotted] & c_j^0 = s(x) \\
\vdots & & \\
c_{j-k} = c_{j-k}^k \arrow[r] & c_{j-k-1}^{k-1} \arrow[uur, dotted] &
\end{tikzcd}$$

using $(*)$.

Conclusion: You do not need to know the B-splines $B_i^k(x)$ to use them!

Diffentiation and integration.

$$\frac{d}{dx} B_i^k(x) = \frac{k}{t_{i+k} - t_i} B_i^{k-1}(x) - \frac{k}{t_{i+k+1} - t_{i+1}} B_{i+1}^{k-1}(x)$$

$$\int_{-\infty}^{x} B_i^k(x) = \left(\frac{t_{i+k+1} - t_i}{k+1}\right) \sum_{j=i}^{\infty} B_j^{k+1}(x)$$

Linear independency:

- $\{B_j, B_{j-1}, \dots, B_{j-k}\}$ are linear independent on (t_j, t_{j+1}))
- $\{B_{-k}^k, \dots \; B_{n-1}^k\}$ are linear independent on (t_0, t_n)