Correction, lecture 02.09.

Given an ODE $y' = f(t,y)$, f satisfy

(L) $$\|f(t,y) - f(t,\tilde{y})\| \leq L \cdot \|y - \tilde{y}\|.$$

and a RK-method

$$k_1 = f(t_n, y_n)$$

$$k_2 = f(t_n + h, y_n + h \cdot k_1)$$

$$y_{n+1} = y_n + \tfrac{1}{2} h (k_1 + k_2)$$

So, the increment function for this method is

$$\Phi(t,y;h) = \tfrac{1}{2}(k_1 + k_2)$$

$$= \tfrac{1}{2}\left(f(t,y) + f(t+h, y + hf(t,y))\right)$$

Q: What is the Lipschitz constant for Φ wrt. y?

$$\|\Phi(t,y;h) - \phi(t,\tilde{y};h)\|$$

$$\overset{(2)}{=} \tfrac{1}{2} \cdot \|f(t,y) - f(t,\tilde{y}) + f(t+h, y + hf(t,y)) - f(t+h, \tilde{y} + hf(t,\tilde{y}))\|$$

$$\overset{(3)}{\leq} \tfrac{1}{2}\big[\|f(t,y) - f(t,\tilde{y})\|$$

$$+ \|f(t+h, y + hf(t,y)) - f(t+h, \tilde{y} + hf(t,\tilde{y}))\|\big]$$

$$\overset{(L)}{\leq} \tfrac{1}{2}\left[L\|y - \tilde{y}\| + L\|y + hf(t,y) - \tilde{y} - hf(t,\tilde{y})\|\right]$$

$$\overset{(3)}{\leq} \tfrac{1}{2}\left[L \cdot \|y - \tilde{y}\| + L\|y - \tilde{y}\| \overset{(2)}{+} hL\|f(t,y) - f(t,\tilde{y})\|\right]$$

$$\overset{(L)}{\leq} (L + \tfrac{1}{2} h L^2)\|y - \tilde{y}\|$$

So, for $h < h_{max}$, Φ is Lipschitz in y, with Lipschitz constant

$$M = L + \tfrac{1}{2} h_{max} L^2.$$