• **Lemma** (additional)

Let $D \subset \mathbb{R}^n$ be convex, and the Jacobian of F satisfy

$$\|J(x) - J(y)\| \leq L\|x - y\|, \quad \forall x, y \in D$$

then

$$\|F(x) - F(y) - J(y)(x-y)\| \leq \frac{L}{2}\|x - y\|^2.$$

Proof:

Let $\varphi : [0,1] \to \mathbb{R}^n$ be given by

$$\varphi(\theta) = F(y + \theta(x-y))$$

$$\Rightarrow \varphi'(\theta) = J(y + \theta(x-y)) \cdot (x-y)$$

$$\|\varphi'(\theta) - \varphi'(0)\| \leq \|(J(y+\theta(x-y)) - J(y))(x-y)\|$$

$$\leq \|J(y+\theta(x-y)) - J(y)\| \cdot \|x-y\|$$

$$\leq \theta \cdot L \cdot \|x-y\|^2$$

Let $\Delta = F(x) - F(y) - J(y)(x-y)$

$$= \varphi(1) - \varphi(0) - \varphi'(0)$$

$$= \int_0^1 (\varphi'(\theta) - \varphi'(0))\,d\theta$$

$$\|\Delta\| \leq \int_0^1 \|\varphi'(\theta) - \varphi'(0)\|\,d\theta \leq L \cdot \|x-y\|^2 \int \theta\,d\theta$$

$$= \frac{1}{2} L \cdot \|x-y\|^2$$