Norms and inner products on $\mathbb{R}^m$

Let $\mathbb{R}^m$ denote the set of all m-dimensional column vectors $x = [x_1, x_2, \ldots, x_m]^T$ with real-number coefficients.

Definition 1.2. *A vector norm on $\mathbb{R}^m$ is a function $\|\cdot\| : \mathbb{R}^m \to \mathbb{R}$ satisfying*

1. $\|x\| \geq 0$ *and* $\|x\| = 0 \Leftrightarrow x = 0$,
2. $\|\alpha x\| = |\alpha|\|x\|$,
3. $\|x + y\| \leq \|x\| + \|y\|$

for all $x, y \in \mathbb{R}^m$ and for all $\alpha \in \mathbb{R}$.

Some common examples of vector norms are:

$$\|x\|_1 = \sum_{i=1}^{m} |x_i| \qquad \text{The } l_1\text{-norm}$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{m} x_i^2} \qquad \text{The } l_2\text{-norm (or Euclidean norm)}$$

$$\|x\|_\infty = \max_{1 \leq i \leq m} |x_i| \qquad \text{The } l_\infty\text{-norm (or the max-norm)}$$

Example 1.3. *If $x = [1.3, -3.5, 2.4]$ then $\|x\|_1 = 7.2$, $\|x\|_2 = 4.4385$ and $\|x\|_\infty = 3.5$.*

Two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on $\mathbb{R}^m$ are *equivalent*, that means there exist two real constants c_1 and c_2 such that for all $x \in \mathbb{R}^m$ one has that

$$c_1\|x\|_a \leq \|x\|_b \leq c_2\|x\|_a.$$

For the norms mentioned above, the following can be proved (do it!):

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq \sqrt{m}\|x\|_2 \leq m\|x\|_\infty.$$

Definition 1.4. *An inner product on $\mathbb{R}^m$ is a function $\langle \cdot, \cdot \rangle : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ satisfying*

1. $\langle x, y \rangle = \langle y, x \rangle$,
2. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$,
3. $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, x \rangle$,
4. $\langle x, x \rangle \geq 0$ *and* $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

for all $x, y, z \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$.

The best known inner product on $\mathbb{R}^m$ is

$$\langle x, y \rangle = x^T y = \sum_{i=1}^{m} x_i y_i,$$

but there are others, as we will see later in the course.
Given an inner product, we can always define a norm by $\|x\|^2 = \langle x, x \rangle$. Such norms satisfies the *Cauchy-Schwarz inequality*:

$$|\langle x, y \rangle| \leq \|x\|\|y\|.$$

Two vectors x and y are *orthogonal* if $\langle x, y \rangle = 0$.