Iterative methods:

Problem: $Ax = b$, $A \in \mathbb{R}^{n\times n}$ nonsingular, $a_{ii} \neq 0$.

$$\Updownarrow$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, 2, \cdots, n$$

Splitting: $A = D - E - F$

$$A = \begin{bmatrix} \ddots & & -F \\ & D & \\ -E & & \ddots \end{bmatrix}$$

Given some initial guess $x^{(0)} \in \mathbb{R}^n$.
Then for $k = 0, 1, 2, 3, \cdots$

Jacobi-iterations:

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \Big(b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} x_j^{(k)} \Big), \quad i = 1, 2, \cdots, n$$

or

$$x^{(k+1)} = D^{-1}\big(b + (E + F)x^{(k)}\big)$$

Gauss-Seidel iterations:

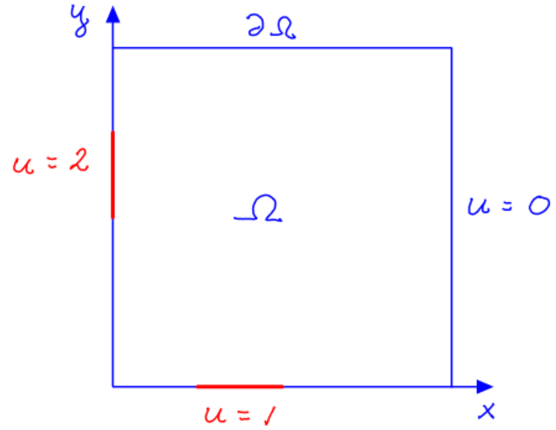
$$x_i^{(k+1)} = \frac{1}{a_{ii}} \Big(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \Big), \quad i = 1, \cdots, n$$

or

$$D x^{(k+1)} = b + E x^{(k+1)} + F x^{(k)}$$

$$\Rightarrow \quad x^{(k+1)} = (D - E)^{-1}\big(b + F x^{(k)}\big)$$

Example:



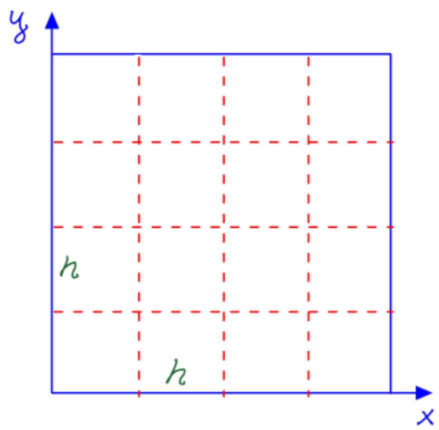
Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{on } \Omega$$

$$u = g \quad \text{on } \partial\Omega$$

(the boundary)

Discretization:



Let $h = 1/N$

$x_i = i \cdot h \quad , \quad i = 0, 1, \cdots, N$

$y_j = j \cdot h \quad , \quad j = 0, 1, \cdots, N$

and

$U_{ij} \approx u(x_i, y_j) \, , \quad i, j = 1, \cdots, N-1$

$U_{0,j}, U_{N,j}, U_{i,0}, U_{i,N}$ are known

Let

$$U_{ij} = \frac{1}{4}\left(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}\right) \quad , \quad i, j = 1, \cdots, N-1$$

or

$$4U_{i,j} - U_{i+1,j} - U_{i-1,j} - U_{i,j+1} - U_{i,j-1} = 0$$

With $N = 3$ we get

$$\left[\begin{array}{ccc:ccc:ccc} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ \hdashline -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ \hdashline 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{array}\right] \left[\begin{array}{c} U_{11} \\ U_{21} \\ U_{31} \\ \hdashline U_{21} \\ U_{22} \\ U_{23} \\ \hdashline U_{31} \\ U_{32} \\ U_{33} \end{array}\right] = \left[\begin{array}{c} U_{10} + U_{01} \\ U_{20} \\ U_{30} + U_{41} \\ \hdashline U_{02} \\ 0 \\ U_{42} \\ \hdashline U_{03} + U_{14} \\ U_{24} \\ U_{34} + U_{43} \end{array}\right]$$

For a general N:

$$A = \begin{bmatrix} T & -I & & & 0 \\ -I & T & -I & & \\ & -I & T & \ddots & \\ & & \ddots & \ddots & -I \\ 0 & & & -I & T \end{bmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}$$

where

$$T = \begin{bmatrix} 4 & -1 & & & 0 \\ -1 & 4 & -1 & & \\ & -1 & 4 & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & & & -1 & 4 \end{bmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}$$

and $I \in \mathbb{R}^{(N-1)\times(N-1)}$ is the identity matrix.

Can prove (TMA4212 Num.diff) that

$$U_{ij} = u(x_i, y_j) + \mathcal{O}\left(\frac{1}{N^2}\right)$$