

Contact during exam:
Anne Kværnø tel. 92663824

Exam in TMA4220
Numerical Solution of Partial Differential Equations Using
Element Methods

Friday December 9, 2011
Time: 15.00 – 19.00

Auxiliary materials: Simple calculator (Hewlett Packard HP30S or Citizen SR-270X)
All printed and hand written material.

Problem 1

Consider the Poisson equation

$$\begin{aligned} -\Delta u &= f & \text{in } \Omega \in \mathbb{R}^2 \\ u &= g & \text{on } \Gamma_D \\ \frac{\partial u}{\partial n} &= \phi & \text{on } \Gamma_N \end{aligned} \tag{1}$$

where $\Gamma_N \cup \Gamma_D = \partial\Omega$ is the boundary of the domain Ω .

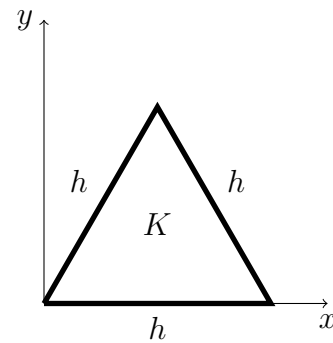
a) Establish the weak formulation:

$$\text{Find } \mathring{u} \in V \text{ such that } a(\mathring{u}, v) = F(v) \quad \forall v \in V \tag{2}$$

corresponding to the problem above. In particular, identify the bilinear form a , the linear form F , the function space V , and explain how the solution u of (1) and the solution \mathring{u} of (2) are related.

Comment: If you are unsure about how to deal with inhomogeneous boundary conditions, change them to something you are comfortable with, for example $u = 0$ on Γ_D , in which case $\mathring{u} = u$. But this simplification gives a slight reduction of the score of this point.

We would like to find an approximation to the solution of (2) by using a finite element method. As a reference element, we choose an equilateral triangle, with side length h . Use linear, nodal element functions, with one node in each corner of the element.



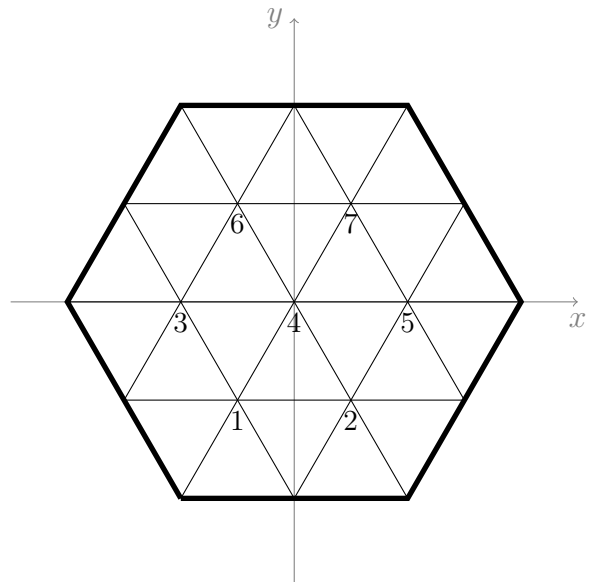
- b) Find the three linear shape functions for the element K . Find the elemental stiffness matrix A^K and the elemental load vector \mathbf{b}^K .

Hint: The area of an equilateral triangle is $h^2\sqrt{3}/4$, and the volume of a pyramid is $BH/3$, where B is the area of the base and H the height.

Now, consider the Poisson equation (1) with $f = 1$ and homogenous Dirichlet boundary conditions

$$u = 0 \quad \text{on} \quad \partial\Omega.$$

The domain Ω is a regular hexagon, and each edge is of length 1. We will solve this problem by the finite element method, using the finite elements from b), and with a triangulation based on equilateral triangles, see the figure.



- c) Show that the finite element system

$$A_h \mathbf{u} = \mathbf{f}_h,$$

where A_h is the stiffness matrix, \mathbf{f}_h the load vector, and \mathbf{u} is a vector of the approximations of the numerical solutions in the nodes, can be written as

$$\begin{pmatrix} \alpha & \beta & \beta & \beta & 0 & 0 & 0 \\ \beta & \alpha & 0 & \beta & \beta & 0 & 0 \\ \beta & 0 & \alpha & \beta & 0 & \beta & 0 \\ \beta & \beta & \beta & \alpha & \beta & \beta & \beta \\ 0 & \beta & 0 & \beta & \alpha & 0 & \beta \\ 0 & 0 & \beta & \beta & 0 & \alpha & \beta \\ 0 & 0 & 0 & \beta & \beta & \beta & \alpha \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{pmatrix} = \begin{pmatrix} \gamma \\ \gamma \\ \gamma \\ \gamma \\ \gamma \\ \gamma \\ \gamma \end{pmatrix}$$

Find α , β and γ .

d) Let us change the boundary conditions of (1) to

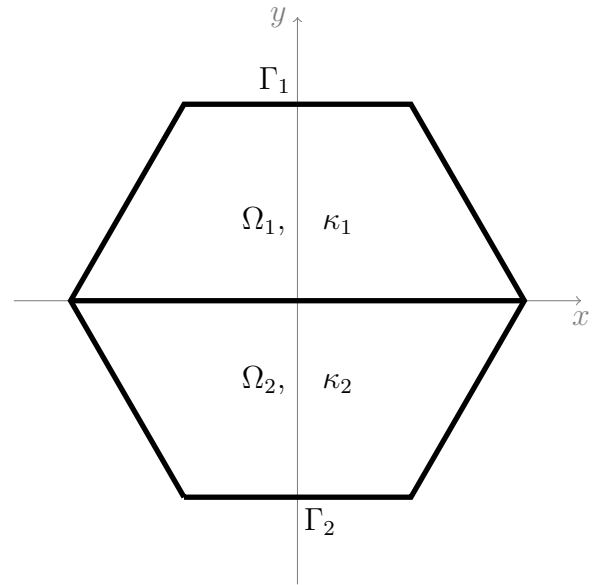
$$u = -x \quad \text{on} \quad \partial\Omega.$$

How will you solve this problem using the finite element method above? Demonstrate the idea by rewriting the first equation (row) of the finite element system to include contributions from the nonhomogeneous boundary conditions.

Consider the Poisson problem again, but this time with domain composed by two subdomains, separated by the x -axis, and with different diffusion constants. More specific, consider

$$\begin{aligned} -\kappa_1 \Delta u &= f && \text{in } \Omega_1 \\ -\kappa_2 \Delta u &= f && \text{in } \Omega_2 \\ u &= 0 && \text{on } \Gamma_1 \\ \frac{\partial u}{\partial n} &= \phi && \text{on } \Gamma_2 \\ \kappa_1 \frac{\partial u}{\partial y} &= \kappa_2 \frac{\partial u}{\partial y} && \text{for } y = 0 \end{aligned}$$

with $\kappa_1, \kappa_2 > 0$.



e) Establish the weak formulation for this problem, and discuss existence and uniqueness of the solution.

Can the weak formulation be reformulated as a minimization problem? If yes, which one? Give a reason for your answer.

Problem 2

Consider the problem

$$0.01 u_{xx} + 2u_x = 0, \quad u(0) = 0, \quad u(1) = 1.$$

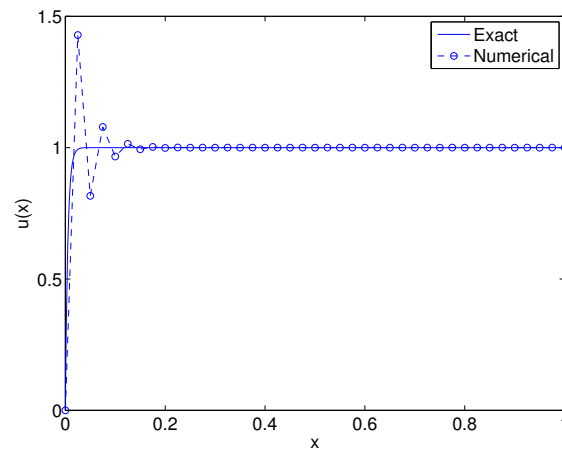
This problem is solved by a linear finite element method with nodal basis functions on a uniform grid, that is $V_h = X_h^1$ with constant $h = 1/M$.

a) Show that the finite element method can be expressed as

$$A_h \mathbf{u} + C_h \mathbf{u} = \mathbf{f}$$

and find the matrices A_h , C_h and the vector \mathbf{f} . Here, A_h represents the diffusion term and C_h the advection term.

The exact and the numerical solution, with $M = 40$ elements is given in the figure below.



b) Explain why the numerical solution oscillates.

How many (equal) elements are needed to avoid oscillations?

c) Explain how we can improve the numerical solution based on linear elements on a uniform grid, without increasing the number of elements.

Problem 3

Point a) and b) are given half weight.

Consider the one-dimensional eigenvalue problem

$$\beta u_x = \lambda u \quad \text{in } \Omega = (0, 1), \quad u(0) = u(1), \quad (3)$$

where β is a given constant. Notice the periodic boundary conditions.

a) Verify that the eigenfunctions u_n and corresponding eigenvalues λ_n ,

$$u_n(x) = e^{ik_n x}, \quad \lambda_n = i\beta k_n, \quad \text{with } k_n = 2\pi n, \quad i = \sqrt{-1}$$

satisfy (3) for $n = 0, \pm 1, \pm 2, \dots$

It can be proved that the weak formulation of the eigenvalue problem (3) is given by:

$$\text{Find } u \in V \text{ and } \lambda \in \mathbb{C} \text{ such that } c(u, v) = \lambda(u, v), \quad \forall v \in V$$

with

$$V = \{v \in H^1(\Omega) \mid v(0) = v(1)\}, \quad c(w, v) = \int_0^1 \beta w_x v \, dx, \quad (w, v) = \int_0^1 wv \, dx.$$

b) Show that the bilinear form c is skew-symmetric, that is

$$c(w, v) = -c(v, w).$$

Following the standard Galerkin method, the discrete eigenvalue problem can be written as

$$\text{Find } u_h \in V_h \subset V \text{ and } \lambda \in \mathbb{C} \text{ such that } c(u_h, v) = \lambda(u_h, v), \quad \forall v \in V_h. \quad (4)$$

c) Let $V_h = X_h^1$ with constant $h = 1/M$. Show that (4) can be written as a linear system of algebraic equations

$$C_h \mathbf{u} = \lambda_h M_h \mathbf{u},$$

and find the matrices C_h and M_h .

In particular, explain how to handle the periodic boundary conditions.