Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods
Høst 2012

Exercise set 1

$\boxed{1}$ Consider the problem

$$-u_{xx} = 1, \quad 0 < x < 1, \quad u(0) = u(1) = 0.$$

a) Derive the exact solution u.
Solution: By integrating twice, and inserting the boundary conditions, we get

$$u(x) = \frac{1}{2}x\,(1-x)\,.$$

b) Show by explicit calculations that

$$\int_0^1 u_x v_x dx = \int_0^1 v dx$$

for all sufficiently smooth v satisfying $v(0) = v(1) = 0$.
Solution: Choose some arbitrary v satisfying the boundary condition: Then

$$\int_0^1 (u_x v_x - v)dx = u_x v|_0^1 - \int_0^1 (u_{xx}+1)\,v dx = 0.$$

c) Compute $J(u)$, where

$$J(v) = \frac{1}{2}\int_0^1 v_x^2 dx - \int_0^1 v dx$$

Solution:

$$J(u) = \frac{1}{2}\int_0^1 \left(\frac{1}{2} - x\right)^2 dx - \int_0^1 \frac{1}{2}x\,(1-x)\,dx = -\frac{1}{24}.$$

d) Let $w_1(x) = a_1 \sin(\pi x)$. Find the value of the amplitude a_1 which minimizes $J(w_1)$. How does a_1 compare with the maximum of the exact solution u?
Solution: By insertion, we get

$$J\left(w_1\right) = \frac{1}{4}a_1^2\pi^2 - \frac{2a_1}{\pi}.$$

So the minimimum is given by

$$\frac{\partial J\left(w_1\right)}{\partial a_1} = \frac{1}{2}a_1\pi^2 - \frac{2}{\pi} = 0, \qquad \Rightarrow \qquad a_1 = \frac{4}{\pi^3}.$$

and

$$\max_{x\in(0,1)} w_1 = a_1 = \frac{4}{\pi^3} \approx 0.129$$

$$\max_{x\in(0,1)} u = u\left(\frac{1}{2}\right) = \frac{1}{8} = 0.125$$

e) Show that $J(w_1) > J(u)$. Is there a big difference?
Solution:

$$J\left(w_1\right) = -\frac{4}{\pi^4} \approx -0.04106 > -\frac{1}{24} \approx -0.041667.$$

f) Let $\varphi_i = \sin\left((2i-1)\pi x\right)$, $i = 1, 2, 3, \ldots$. These functions are infinitely differentiable, and they all satisfy $\varphi_i(0) = \varphi_i(1) = 0$. Compute

$$a_{ij} = \int_0^1 \varphi_j' \varphi_i' dx \quad \text{and} \quad b_i = \int_0^1 \varphi_i dx.$$

Solution: We get

$$a_{ij} = \begin{cases} \frac{\pi^2}{2}\left(2i-1\right)^2 & \textit{for } i = j \\ 0 & \textit{otherwise} \end{cases}, \quad b_i = \frac{2}{\left(2i-1\right)\pi}$$

g) Let $V_N = \operatorname{span}\left\{\varphi_1, \varphi_2, \ldots, \varphi_N\right\}$. Set up and solve the problem

$$\text{Find } w_N \in V_N \text{ such that } \int_0^1 w_{N,x} v_x dx = \int_0^1 v dx, \quad \forall v \in V_N.$$

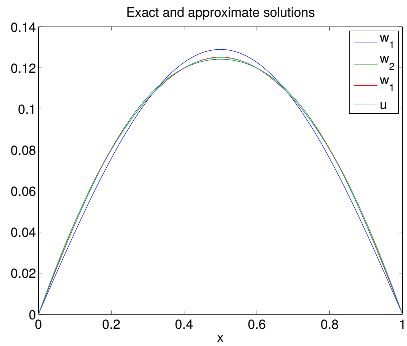
Solution: Choose $w_N = \sum_{j=1}^N \hat{w}_i \varphi_i$ *such that:*

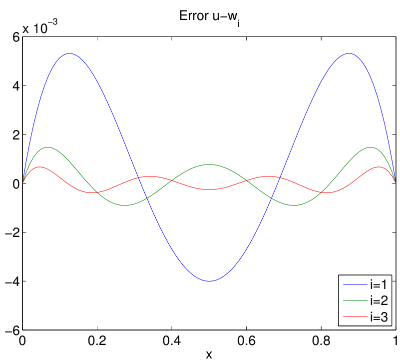
$$\sum_{j=1}^N \hat{w}_j \int_0^1 \phi_j' \phi_i' dx = \int_0^1 \phi_i dx, \quad i = 1, \ldots, N$$

or simply $A\hat{\mathbf{w}} = \mathbf{b}$ *where the elements of* A *and* b *are given in* ***d)****. But* A *is a diagonal matrix, so we get*

$$\hat{w}_i = \frac{b_i}{a_{ii}} = \frac{4}{\pi^3\left(2i-1\right)^3}, \qquad \textit{and} \qquad w_N = \sum_{i=1}^N \frac{4}{\pi^3\left(2i-1\right)^3} \sin\left(\left(2i-1\right)\pi x\right).$$

h) Plot the error $u - w_N$ for $N = 1, 2, 3$.
Solution:





$\boxed{2}$ Given the weak statement:

$$\text{Find } u \in V \quad \text{s.t.} \quad a(u,v) = F(v), \qquad \forall v \in V \tag{1}$$

and the minimization principle:

$$u = \arg\min_{u \in V} J(v), \qquad \text{with} \quad J(v) = \frac{1}{2}a(v,v) - F(v). \tag{2}$$

a) Show that (1) and (2) are equivalent whenever a is bilinear, symmetric and positive definite, and F is linear. State clearly which properties you are using in you arguments.

Solution: The principle(2) implies that $J(u) \leq J(u+v)$, $\forall v \in V$ (V is a linear space).

$$\begin{aligned}
J(u+v) &= \frac{1}{2}a(u+v,u+v) - F(u+v) && (\textit{Definition})\\
&= \frac{1}{2}a(u,u) - F(u) + \frac{1}{2}(a(u,v) + a(v,u)) - F(v) + \frac{1}{2}a(v,v) && (\textit{Linearity})\\
&= J(u) + (a(u,v) - F(v)) + \frac{1}{2}a(v,v) && (\textit{Symmetry})\\
&> J(u) + (a(u,v) - F(v)), \qquad \forall v \neq 0 && (\textit{Positivity})
\end{aligned}$$

So clearly, if (1) is satisfied, then u is a minimizer of J and vice versa.

b) Take $V = \mathbb{R}^n$, and just show, by appropriate choice of a and F, that the minimizer $u \in V$ of $J(v) = \frac{1}{2}v^TGv - v^Tb$ for any symmetric, positive definite matrix $G \in \mathbb{R}^{n\times n}$ and $v \in \mathbb{R}^n$ satisfies $Gu = b$.
*Solution: Let $w \in \mathbb{R}^n$. Set $a(w,v) = w^TGv$ and $F(v) = v^Tb$. Cleary, a is bilinear ($a(w_1+w_2,v) = (w_1+w_2)^TGv = w_1Gv + w_2Gv = a(w_1,v) + a(w_2,v)$, etc. Further a is symmetric since $a(w,v) = w^TGv = w^TG^Tv = v^TGw = a(v,w)$, and finally a is positive definite since $a(v,v) = v^TGv > 0$ for all $v \neq 0$. So the result of point **a)** apply, the minimizer u satisfies (1), that is*

$$\begin{aligned}
u^TGv &= v^Tb, \qquad \forall v \in \mathbb{R}^n\\
v^TGu &= v^Tb, \qquad \forall v \in \mathbb{R}^n\\
v^T(Gu-b) &= 0, \qquad \forall v \in \mathbb{R}^n
\end{aligned}$$

which is satisfied if and only if

$$Gu = b.$$

$\boxed{3}$ Write a code for solving the equation

$$-u_{xx} = x^4, \quad 0 < x < 1, \quad u(0) = 0, \quad u(1) = 0.$$

using the finite element method with equidistant grid ($x_i = ih$, $h = 1/N$), and the basis functions

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h}, & \text{for } x_{i-1} \leq x \leq x_i,\\ \frac{x_{i+1} - x}{h} & \text{for } x_i \leq x \leq x_{i+1},\\ 0 & \text{otherwise.} \end{cases}$$

for $i = 0, 1, 2, \ldots, N$. Compare the numerical solution with the exact solution, and plot the error.

Solution: Because of the Dirichlet (essential) boundary conditions, we do not include the basis functions φ_0 and φ_N. We get

$$\varphi_i'(x) = \begin{cases} \frac{1}{h} & \text{for } x_{i-1} < x < x_i, \\ -\frac{1}{h} & \text{for } x_i < x < x_{i+1}, \\ 0 & \text{otherwise} \end{cases}$$

that is

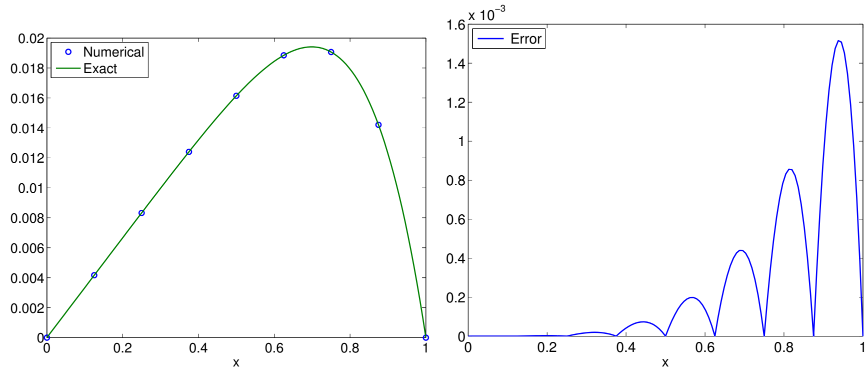
$$a_{ij} = \int_0^1 \varphi_j' \varphi_i' dx = \begin{cases} \frac{2}{h}, & i = j,\ i \neq N, \\ -\frac{1}{h}, & i = j+1 \text{ or } i = j-1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} b_i = \int_0^1 x^4 \varphi_i dx &= \frac{x_i^6 - x_{i-1}^6}{6h} - \frac{x_{i-1}(x_i^5 - x_{i-1}^5)}{5h} - \frac{x_{i+1}^6 - x_i^6}{6h} + \frac{x_{i+1}(x_{i+1}^5 - x_i^5)}{5h} \\ &= -\frac{x_{i+1}^6 - 2x_i^6 + x_{i-1}^6}{6h} + \frac{x_{i+1}^6 - (x_{i-1} + x_{i+1})x_i^5 + x_{i-1}^6}{5h} \end{aligned}$$

Solve the system $A\mathbf{u} = b$, and the numerical solution is given by $u_h(x) = \sum_{i=1}^N u_i \varphi_i(x)$. Notice that $u_h(x_i) = u_i$, and u_h is piecewise linear, which makes it very easy to plot this function. The exact solution for this problem is $u(x) = x(1 - x^5)/30$.

The plot of the solutions as well as the error is given below.



The MATLAB code for solving the problem is given in Figure 1.

```
N = 8;
h = 1/N;
x = linspace(0,1,N+1)';

A=(2/h)*diag(ones(N-1,1))-(1/h)*(diag(ones(N-2,1),1)+diag(ones(N-2,1),-1));

i = 2:N;
b = -(x(i+1).^6-2*x(i).^6+x(i-1).^6)/(6*h) ...
    + (x(i+1).^6 - (x(i-1)+x(i+1)).*x(i).^5 + x(i-1).^6)/(5*h);

u = A\b;
```

Figure 1: MATLAB code for problem 3.