Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods
Høst 2012

Exercise set 2

1 If you are not familiar with the the Lebesgue spaces $L^p(\Omega)$ and the Sobolov spaces $H^p(\Omega)$, you should read section 2.3.1 and 2.4.0-2.4.2.

True or false:

a) The set $S = \left\{v \in C^0(0,1) \ : \ v\left(\frac{1}{2}\right) = 1\right\}$ is a linear (vector) space.
Solution: False. If $v_1, v_2 \in S$, *then* $v_1 + v_2 \notin S$ *since* $(v_1+v_2)\left(\frac{1}{2}\right) = 1$.

b) For $X = H_0^1\left((0,1)\right)$, $L(v) = \int_0^1 xvdx$ is a linear functional.
Solution: True: $L : X \to \mathbb{R}$ *and* $L(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 L(v_1) + \alpha_2 L(v_2)$, $\forall \alpha_1, \alpha_2 \in \mathbb{R}$ *and* $v_1, v_2 \in X$.

c) For $Z = \mathbb{R}$, $(x,y)_Z = |x|\,|y|$ is a valid inner product.
Solution: False: $(x,y)_Z$ *is not bilinear. E.g* $(x_1+x_2, y)_Z = |x_1+x_2|\,|y| \neq |x_1|\,|y| + |x_2|\,|y|$ *if* x_1 *and* x_2 *has opposite signs.*

d) The only v in $H^1(\Omega)$ for which $|w|_{H^1(\Omega)}$ (the H^1 semi-norm) is zero is $v = 0$.
Solution: False: Try $w = 1$.

e) The function $v = x^{3/4}$ is in $L^2\left((0,1)\right)$; in $H^1((0,1))$; in $H^2\left((0,1)\right)$.
Solution: We have

$$\|v\|^2_{L^2(\Omega)} = \int_0^1 \left(x^{3/2}\right)^2 dx = \frac{2}{3} < \infty$$
$$\|v_x\|^2_{L^2(\Omega)} = \int_0^1 v_x^2 dx = \frac{9}{8} < \infty$$
$$\|v_{xx}\|_{L^2(\Omega)} = \int_0^1 v_{xx}^2 dx \not< \infty$$

So $v \in L_2\left((0,1)\right)$ *and* $v \in H^1\left((0,1)\right)$, *but not in* $H^2\left((0,1)\right)$.

f) For $v = e^{-10x}$, $|v|_{H^2((0,1))} = |v|_{H^1((0,1))}$.
Solution: False.

2 Consider the fourth-order problem:

$$u_{xxxx} = f \quad \text{in } \Omega = (0,1),$$
$$u(0) = u_x(0) = u(1) = u_x(1) = 0.$$

This “biharmonic” equation is relevant to, amongst other applications, the bending of beams.

a) Find a symmetric, positive form a over V and a linear form F such that the solution u of the equation satisfies

$$a(u,v) = F(v),\ \forall v \in V.$$

Solution: As always, choose a test function $v \in V$, integrate over the domain, and apply partial integration:

$$\int_0^1 u_{xxxx} v dx = u_{xxx} v|_0^1 - \int u_{xxx} v_x dx = u_{xxx} v|_0^1 - u_{xx} v_x|_0^1 + \int_0^1 u_{xx} v_{xx} dx = \int_0^1 f v dx.$$

The two boundary terms disappear if $v(0) = v(1) = v_x(0) = v_x(1) = 0$, so we can use $a(u,v) = \int_0^1 u_{xx} v_{xx} dx$ and $F(v) = \int_0^1 f v dx$. Further, a is clearly symmetric, and it is positive definite since $a(v,v) = \int_0^1 v_{xx}^2 dx \geq 0$, and $a(v,v) = 0$ only if $v_{xx} \equiv 0$ which implies $v \equiv 0$ because of the boundary conditions.

b) How should V be defined?
Solution: $V = H_0^2((0,1)) = \{v \in H^2((0,1)) : v(0) = v(1) = v_x(0) = v_x(1) = 0\}$.

c) Do you think that $F(v) = v_x\left(\frac{1}{2}\right)$ is a linear, bounded functional on V?
Solution: Yes, since

$$F(v) = v_x\left(\frac{1}{2}\right) = \int_0^{1/2} v_{xx}(x)\, dx \qquad \textit{since } v_x(0) = 0$$

$$|F(v)| \leq \int_0^{1/2} |v_{xx}(x)| \cdot 1 dx \leq \left(\int_0^{1/2} v_{xx}^2 dx\right)^{1/2} \left(\int_0^{1/2} 1^2 dx\right)^{1/2} \qquad \textit{Cauchy-Schwartz}$$

$$= \frac{\sqrt{2}}{2}\left(\int_0^{1/2} v_{xx}^2 dx\right)^{1/2} \leq \frac{\sqrt{2}}{2}\left(\int_0^{1} v_{xx}^2 dx\right)^{1/2}$$

$$= \frac{\sqrt{2}}{2} |v|_{H^2((0,1))} \leq \frac{\sqrt{2}}{2} \|v\|_{H^2((0,1))}.$$

3 Consider the problem with a discontinuous jump in conductivities:

$$-\kappa^L u_{xx}^L = f^L, \qquad 0 < x < \frac{1}{2},$$
$$-\kappa^R u_{xx}^R = f^R, \qquad \frac{1}{2} < x < 1,$$

with boundary conditions

$$u^L(0) = 0, \qquad u^R(1) = 0,$$
$$u^L\left(\tfrac{1}{2}\right) = u^R\left(\tfrac{1}{2}\right),$$
$$\kappa^L u_x^L\left(\tfrac{1}{2}\right) = \kappa^R u_x^R\left(\tfrac{1}{2}\right), \qquad \text{(continuity of flux)}.$$

Here, κ^L and κ^R are strictly positive. Let $V = \{v \in H^1((0,1)) : v(0) = v(1) = 0\}$. Find a and F such that the solution u satisfies

$$a(u,v) = F(v), \qquad \forall v \in V.$$

Solution: Choose some $v \in V$. Then

$$
\begin{aligned}
-\int_0^{1/2} \kappa^L u^L_{xx} v dx - \int_{1/2}^1 \kappa^R u^R_{xx} v dx &= -\kappa^L u_x v|_0^{1/2} + \int_0^{1/2} \kappa^L u^L_x v_x dx - \kappa^R u^R_x v|_{1/2}^1 + \int_{1/2}^1 \kappa^R u^R_x v dx \\
&= \left(-\kappa^L u^L_x\left(\frac{1}{2}\right) + \kappa^R u^R_x\left(\frac{1}{2}\right)\right) v\left(\frac{1}{2}\right) + \int_0^{1/2} \kappa^L u^L_x v_x dx + \int_{1/2}^1 \kappa^R u^R_x v_x dx \\
&= \int_0^{1/2} f^L v dx + \int_{1/2}^1 f^R v dx
\end{aligned}
$$

The first term of the second line is zero because of the continuity of flux condition. So we can choose

$$
a(u,v) = \int_0^{1/2} \kappa^L u^L_x v_x dx + \int_{1/2}^1 \kappa^R u^R_x v_x dx, \qquad F(v) = \int_0^{1/2} f^R v dx + \int_{1/2}^1 f^L v dx.
$$

4 Given the Helmholtz problem

$$
\begin{aligned}
-u_{xx} + \sigma u &= f \text{ on } (0,1)\,, \\
u(0) = u(1) &= 0.
\end{aligned}
$$

where $\sigma > 0$ is a constant. Set up the weak form for this problem. Show that, when this problem is solved by a Galerkin method, using $V_h = \text{span}\,\{\phi_i\}_{i=1}^N$, the discrete problem can be written as

$$
(A + \sigma M)\mathbf{u} = \mathbf{f}.
$$

Set up the matrix M for $V_h = X_h^1$ on a uniform grid.

Solution: Multiply by the equation by a test function v, integrate over the domain $(0,1)$, use partial integration and get rid of the boundary terms by require $v(0) = v(1) = 0$. The the weak formualtion becomes:

$$
\textit{Find } u \in H_0^1(\Omega) \textit{ such that } \int_0^1 v_x u_x dx + \sigma \int_0^1 uv dx = \int_0^1 fv dx, \quad \forall v \in H_0^1(0,1).
$$

Choose $V_h = span\{\varphi_1, \varphi_2, \dots, \varphi_N\}$, let our unknown approximation be written as

$$
u_h(x) = \sum_{j=1}^N u_j\, \varphi_j(x),
$$

where the coefficients u_j is found from

$$
\sum_{j=1}^N \int_0^1 \left(\frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx}\right) dx + \sigma \sum_{j=1}^N \int_0^1 (\varphi_j\, \varphi_i) dx = \int_0^1 (f\, \varphi_i) dx, \qquad i = 1, 2, \cdots, N.
$$

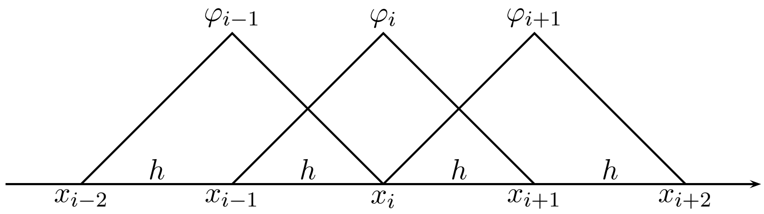
This is a linear system of equations

$$
A\mathbf{u} + \sigma M \mathbf{u} = \mathbf{f}
$$

where

$$(A_{i,j} = \int_0^1 \left(\frac{d\varphi_j}{dx}\,\frac{d\varphi_i}{dx}\right) dx, \qquad M_{i,j} = \int_0^1 (\varphi_j\,\varphi_i) dx \quad \textit{and} \quad f_i = \int_0^1 (f\varphi_i) dx.$$

Let $V_h = X_h^1$ *on a uniform grid, so that* $h = 1/N$ *and* $x_i = ih$, $i = 0, 1, \ldots, N$.



The non-zeros elements of the stiffness matrix A *and the mass matrix* M *is*

$$A_{i,i} = \int_{x_{i-1}}^{x_{i+1}} \left(\frac{d\varphi_i}{dx}\right) dx = \frac{2}{h}, \qquad A_{i,i+1} = \int_{x_i}^{x_{i+1}} \frac{d\varphi_{i+1}}{dx}\frac{d\varphi_i}{dx} dx = -\frac{1}{h} = A_{i+1,i}$$

$$M_{i,i} = \int_{x_{i-1}}^{x_{i+1}} \varphi_i^2 dx = \frac{2}{3}h, \qquad M_{i,i+1} = \int_{x_i}^{x_{i+1}} \varphi_{i+1}\varphi_i dx = \frac{1}{6}h = M_{i+1,i}.$$

All the other elements are 0. So

$$M = \frac{h}{6}\begin{pmatrix} 4 & 1 & & & 0 \\ 1 & 4 & 1 & & \\ & 1 & 4 & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & 4 \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}.$$