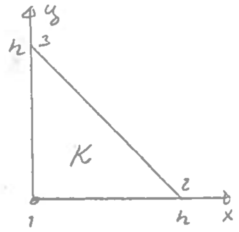
Solution Exercise set 4.

1. $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, d\Omega$

a)



Shape functions:

$$\varphi_1^K = 1 - \frac{x}{h} - \frac{y}{h}$$

$$\varphi_2^K = \frac{x}{h}$$

$$\varphi_3^K = \frac{y}{h}$$

The elements of A_h^K is

$$(A_h^K)_{\alpha\beta} = \int_K \left(\frac{\partial \varphi_\alpha^K}{\partial x} \frac{\partial \varphi_\beta^K}{\partial x} + \frac{\partial \varphi_\alpha^K}{\partial y} \frac{\partial \varphi_\beta^K}{\partial y} \right) d\Omega$$

so that

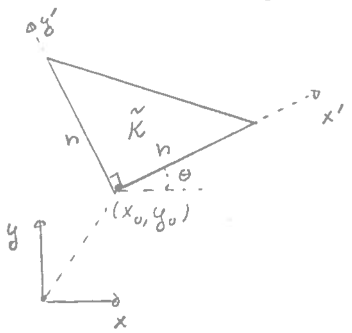
$$(A_h^K)_{11} = \int_0^h \int_0^{h-y} \left[\left(-\frac{1}{h}\right)^2 + \left(-\frac{1}{h}\right)^2 \right] dx\,dy = 1$$

$$(A_h^K)_{12} = \int_0^h \int_0^{h-y} \left[\left(-\frac{1}{h}\right) \cdot \frac{1}{h} \right] dx\,dy = -\frac{1}{2}$$

etc. So

$$A_h^K = \frac{1}{2} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

b)



Consider the element from a), translated to some arbitrary point (x_0, y_0), and rotated an arbitrary angle θ. But the size and the shape is kept.

As before,

$$(A_h^{\tilde{K}})_{\alpha\beta} = \int_{\tilde{K}} \nabla \varphi_\alpha^{\tilde{K}} \cdot \nabla \varphi_\beta^{\tilde{K}} \, d\Omega$$

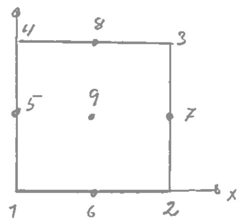
But the integrand $\nabla \varphi_\alpha^{\tilde{K}} \cdot \nabla \varphi_\beta^{\tilde{K}}$ represent a dot product, which we know is geometric invariant in the sense that the dot product is unaffected by the choice of Cartesian coordinate system. Hence we introduce another Cartesian system (x', y') as shown in the figure. In this coordinate system, the shape functions are

$$\varphi_1^{\tilde{K}} = 1 - \frac{x'}{h} - \frac{y'}{h}, \quad \varphi_2^{\tilde{K}} = \frac{x'}{h}, \quad \varphi_3^{\tilde{K}} = \frac{y'}{h}$$

so

$$A_h^{\tilde{K}} = \frac{1}{2} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$ as expected.

3a) We have the following FE:



The definition of a finite element is given in B&S, def. 3.1.1., alternatively Q, sec. 4.4.1

i) $K = [0,1] \times [0,1]$

ii) $P = Q_2 = \{ \sum_j c_j p_j(\underline{x}) q_j(\underline{x}), \quad p_j, q_j \in \mathbb{P}_2 \}.$ (B&S p. 85)

iii) $\mathcal{N} = \{N_1, N_2, \cdots, N_9\}$, where, in our case

$N_i v = v(\underline{x}_i)$, $i = 1, 2, \cdots, 9$, and $\underline{x}_i$ are the nodes

as shown in the figure.

What we have to prove is that $\mathcal{N}$ really is a basis for P', or, said differently, given the values of v in the nodes, $v \in P$ is defined uniquely. To do so, we can use Lemma 3.1.4 in B&S, that is, prove

Given $v \in Q_2$ with $v(\underline{x}_i) = 0$, $i = 1, 2, \cdots, 9$ then $v \equiv 0$

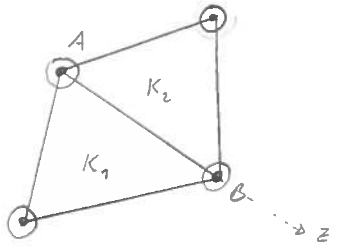
So, we assume $v(\underline{x}_i) = 0$ for $i = 1, 2, \cdots, 9$.
Clearly, let $\underline{x} = (x, y)$ and
$p_0(x) = v(x, 0) \in \mathbb{P}_2$, is a second order polynomial which is zero for $x = 0$, ½ and 1, 3 distinct points. Thus $p_0(x) \equiv 0$
Similar for $p_1(x) = v(x, 1/2) \equiv 0$ and $p_2(x) = v(x, 1) \equiv 0$.

Let $(\bar{x}, \bar{y})$ be some arbitrary point in K. Let $g(y) = v(\bar{x}, y)$, Then $g(y)$ is a second order polynomial in y, which is zero for $y = 0$, ½ and 1. Thus $g(y) \equiv 0$ and in particular,

$g(\bar{y}) = v(\bar{x}, \bar{y}) = 0$

which means $v \equiv 0$ since $(\bar{x}, \bar{y})$ is arbitrary chosen.

3b)

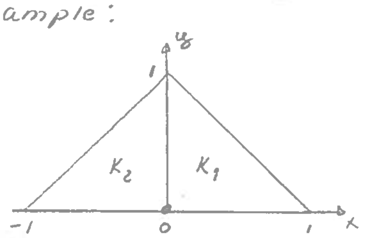


In this case, $P = P_3$. This means that a function $v \in P_3$ is a one-dimensional cubic polynomial on the line AB, call it $p(z)$. This polynomial satisfies

$$p(A),\ \left.\frac{dp}{dz}\right|_{z=A},\ p(B),\ \left.\frac{dp}{dz}\right|_{z=B} \quad \text{known.}$$

Thus, $p(z)$ is the one-dimensional hermite polynomial, which is unique. Thus $\varphi_1^{K_1}|_{AB} = \varphi^{K_2}|_{AB}$ and $\varphi \in C^0$.

But it is not C^1, which can be proved by the following example:



The shape function $\varphi_1^{K_1}$, corresponding to the node (0,0) (that is $\varphi_0^{K_1} = 1$, $\frac{\partial \varphi_0^{K_1}}{\partial x} = \frac{\partial \varphi_0^{K_1}}{\partial y} = 0$ at (0,0)) is, according to Maple

$$\varphi_0^{K_1}(x,y) = 1 - 3x^2 - 13xy - 3y^2 + 2x^3 + 13x^2y + 13xy^2 + 2y^3$$

(check it). The shape function at K_2, corresponding to the same node is

$$\varphi_0^{K_2}(x,y) = \varphi_0^{K_1}(-x,y)$$

But $\left.\frac{\partial \varphi_0^{K_1}}{\partial x}\right|_{x=0} = 13y(y-1)$ and $\left.\frac{\partial \varphi_0^{K_2}}{\partial x}\right|_{x=0} = -13y(y-1)$

so $\frac{\partial \varphi_0}{\partial x}$ is discontinuous over the edge, and $\varphi_0 \notin C^1$.