Exercise 5, problem 2 (Q, ch. 5, ex. 2)

(1) $u_t - (\alpha u_x)_x - \beta u = 0$ on $0<x<1$, $t>0$

$u = u_0(x)$ for $t=0$

$u = \eta$ for $x=0$

$\alpha u_x + \gamma u = 0$ for $x=1$

Parameters: $\alpha = \alpha(x)$, $\beta, \gamma, \eta \in \mathbb{R}$ (constants), $\beta > 0$

For simplicity, assume $\eta = 0$ in the following.
The generalization to $\eta \neq 0$ is trivial, and left to you.

a) Weak formulation:

For each $t>0$ find $u(t) \in V$ s.t

$$\int_0^1 u_t \cdot v \, dx + a(u(t), v) = 0, \quad \forall v \in V$$

where $V = \{v \in H^1(0,1);\ v(0) = 0\}$
and

$$a(u,v) = \int_0^1 \alpha \cdot u_x \cdot v_x \, dx - \beta \int_0^1 u \cdot v \, dx + \gamma u(1) \cdot v(1)$$

and with $u(0) = u_0$.

This has a unique solution (Q, p. 120-121) if

$u_0 \in L^2(\Omega)$ and a is bilinear, continuous and weakly coercive:

Bilinear: Obvious
Continuous: $\exists M > 0$ s.t. $|a(u,v)| \leq M \cdot \|u\|_V \cdot \|v\|_V$, $\forall u, v \in V$
We have

$$|a(u,v)| \leq \left| \int_0^1 \alpha u_x v_x \, dx \right| + |\beta| \left| \int_0^1 u \cdot v \, dx \right| + |\gamma| \cdot |u(1)| \cdot |v(1)|.$$

Here, let $\alpha_1 = \max_{0<x<1} |\alpha(x)|$

$$\left| \int_0^1 \alpha u_x v_x \, dx \right| \leq \alpha_1 \cdot \int_0^1 u_x \cdot v_x \, dx \leq \alpha_1 \cdot \left(\int_0^1 u_x^2 \, dx \right)^{1/2} \cdot \left(\int_0^1 v_x^2 \, dx \right)^{1/2}$$

$$\leq \alpha_1 \cdot \|u\|_V \cdot \|v\|_V$$

similar:

$$\left| \int_0^1 u \cdot v \, dx \right| \leq \|u\|_V \cdot \|v\|_V$$

And

$$|u(1)| = \left| \int_0^1 u_x \cdot 1 \, dx \right| \leq \left(\int_0^1 u_x^2 \, dx \right)^{1/2} \left(\int_0^1 1^2 \, dx \right)^{1/2} \leq \|u\|_V$$

using $u(0) = 0$.

We conclude that

$$|a(u,v)| \leq (\alpha_1 + |\beta| + |\gamma|) \cdot \|u\|_V \cdot \|v\|_V \quad , \forall u, v \in V$$

if $|\alpha| < \alpha_0$ $\quad (\alpha \in L^\infty(0,1))$

<u>Weak coercivity:</u> We assume that $\alpha \geq \alpha_0 > 0$, on $(0,1)$ and $\gamma > 0$. Then

$$a(v,v) = \int_0^1 \alpha v_x^2 dx - \beta \int_0^1 v^2 dx + \gamma v(1)^2$$

$$\geq \alpha_0 \int v_x^2 dx - \beta \int_0^1 v^2 dx$$

$$= \alpha_0 \int_0^1 (v_x^2 + v^2) dx - (\alpha_0 + \beta) \int_0^1 v^2 dx$$

$$= \alpha_0 \|v\|_V^2 - (\alpha_0 + \beta) \|v\|^2_{L^2(0,1)}$$

Thus by choosing $\lambda > \alpha_0 + \beta$ the weak coercivity condition is ~~proved~~ satisfied, and we have a unique solution.

b) The semidiscrete problem is

$$\int \frac{\partial}{\partial t} u_h \cdot v_h \, dx + a(u_h, v_h) = 0 \quad , \forall v_h \in V_h$$

and $u_h = u_h(t)$.

Let $v_h = u_h(t)$ (for a fixed t). Then

$$\int \frac{\partial}{\partial t} u_h u_h \, dx = \frac{1}{2} \frac{d}{dt} \|u_h\|_{L^2(0,1)}$$

Further

$$a(u_h, u_h) = \int_0^1 \alpha u_{h,x}^2 \, dx - \beta \int_0^1 u_h^2 \, dx + \gamma u_{N+1}^2$$

assuming Lagrangian basis functions, so

$$\frac{1}{2} \frac{\partial}{\partial t} \|u_h\|^2_{L^2(0,1)} + \alpha_0 \|u_{h,x}\|^2_{L^2(0,1)} \leq \beta \cdot \|u_h\|^2_{L^2(\Omega)} - \gamma u_{N+1}^2$$

$$\leq \beta \|u_h\|^2_{L^2(\Omega)} \quad \text{if } \gamma \geq 0$$

Since $\alpha_0 \|u_{h,x}\|^2_{L^2(0,1)} > 0$ we get

$$\|u_h(t)\|^2_{L^2(0,1)} \leq \|u_h(0)\|^2_{L^2(0,1)} \cdot e^{2\beta t} \quad , t \geq 0$$

(There may be better solutions!)

c) Let $u_h(t) = \sum_{i=1}^{N+1} u_i(t) \cdot \varphi_i(x)$

Then the discrete formulation is

$$M \cdot \frac{\partial \underline{u}}{\partial t} + A\underline{u} - \beta M \underline{u} = 0$$

where $M_{ij} = \int_0^1 \varphi_i \varphi_j \, dx$, $A_{ij} = \int_0^1 \alpha(x) \varphi_{i,x} \varphi_{j,x} \, dx$

The explicit Euler is now

$$M \frac{\underline{u}^{k+1} - \underline{u}^k}{\Delta t} + A\underline{u}^k - \beta M \underline{u}^k = 0$$

or

$$\underline{u}^{k+1} = \underline{u}^k - \Delta t M^{-1} A \underline{u}^k + \Delta t \beta \underline{u}^k = C \cdot \underline{u}^k$$

We have not really discussed stability in this case. Since $\beta > 0$, we assume some growth. In fact, we will assume this to be stable if

$$|\lambda(C)| \leq 1 + \Delta t \cdot \varkappa \qquad \text{for some } \varkappa > 0.$$

And this is true if $0 < \Delta t < -2\lambda_i$,

where λ_i are the eigenvalues of $M^{-1}A$, and $\varkappa = \beta$.