Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods
Fall 2012

Exercise set 5

1 Given the equation:

$$u_t = u_{xx} + \beta u, \qquad 0 < x < 1$$
$$\frac{\partial u}{\partial n}(0,t) = 0, \quad u(1,t) = 0,$$
$$u(x,0) = \cos\left(\frac{\pi}{2}x\right),$$

and β is some constant.

a) Derive the exact solution for the equation.
Solution:

$$u(x,t) = e^{\left(-\frac{\pi^2}{4}+\beta\right)t}\cos\left(\frac{\pi}{2}x\right)$$

b) Set up the weak formulation of the problem.
Solution: Multiply the equation by a test function $v(x)$, integrate over $\Omega = (0,1)$:

$$\int_0^1 \frac{\partial u}{\partial t} v dx = -\int_0^1 u_x v_x dx + |_0^1 u_x v + \beta \int_0^1 uv dx$$

Let $V = \{v \in H^1(0,1) \;:\; v(1) = 0\}$. Then we get

For each $t > 0$ find $u(t) \in V$ such that $$\int_0^1 \frac{\partial u}{\partial t} v dx + a(u,v) = 0, \qquad \forall v \in V,$$

where $a(u,v) = \int_0^1 u_x v_x dx - \beta \int_0^1 uv dx$.

c) Write a MATLAB code to solve this problem. In space, use $V_h = X_h^1$ and a uniform grid. If time, try all three schemes: Forward and backward Euler, as well as Crank-Nicolson. Experiment with different stepsizes, and compare your numerical results with the exact solution.
Solution: The FEM solution, using $V_h = X_h^1$ with constants stepsize $h = 1/N$ becomes

$$M_h \frac{\partial \mathbf{u}_h}{\partial t} = -A_u \mathbf{u}_h + \beta M_h \mathbf{u}_h$$

with

$$M_h = \frac{h}{6}\begin{pmatrix} 2 & 1 & & \\ 1 & 4 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 4 \end{pmatrix}, \qquad A_h = \frac{1}{h}\begin{pmatrix} 1 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}$$

One step of Crank-Nicolsons method becomes

$$M_h \frac{1}{\Delta t}(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n) = \frac{1}{2}(-A_h + \beta M_h)(\mathbf{u}_h^{n+1} + \mathbf{u}_h^n).$$

The matlab coding and experimentation is left for you.

2 Quarteroni Chapter 5, Exercise 2.
In b), no convergence analysis is required.

3 For those of you who have taken the course Numerical Mathematics or something equivalent:

Write down the set of fully discrete equations in the case of solving the semidiscretized system

$$M_h \dot{\mathbf{u}}(t) + A_h \mathbf{u}(t) = \mathbf{f}(t)$$

(Q: p.121, last line), by

a) A second order Adams-Bashforth scheme
b) A second order Adams-Moulton scheme
c) A second order Backward-Differentiation scheme

Solution: For simplicity, let $\mathbf{F}_n = A_h \mathbf{u}_h^n + \mathbf{f}(t_n)$, *where* $t_n = t_0 + n\Delta t$.
Adams-Bashforth:

$$M_h(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n) = \frac{\Delta t}{2}\left(3\mathbf{F}_n - \mathbf{F}_{n-1}\right).$$

Adams-Moulton (coincide with the Crank-Nicolson scheme)

$$M_h(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n) = \frac{\Delta t}{2}\left(\mathbf{F}_{n+1} + \mathbf{F}_n\right).$$

BDF:

$$M_h\left(\frac{3}{2}\mathbf{u}_h^{n+1} - 2\mathbf{u}_h^n + \frac{1}{2}\mathbf{u}_h^{n-1}\right) = \Delta t \mathbf{F}_{n+1}.$$

4 Problem 1-6 in the note *Spectra of the continuous and discrete Laplace operator* by Einar Rønquist.
Solution:

1. *From the definition of eigenvectors and eigenvalues, we have that*

$$M_h \mathbf{u}_j = \lambda_j(M_h)\mathbf{u}_j, \qquad A_h \mathbf{u}_j = \lambda_j(A_h)\mathbf{u}_j,$$

where $\lambda_j(\cdot)$ *are the eigenvalues of* M_h *and* A_h *respectively. We know that the eigenvectors are the same in this case (not in general true!). But then*

$$M_h^{-1} A_h \mathbf{u}_j = \lambda_j(A_h) M_h^{-1}\mathbf{u}_j = \frac{\lambda_j(A_h)}{\lambda_j(M_h)}\mathbf{u}_j,$$

so $\lambda_j(M_h^{-1}A_h) = \lambda_j(A_h)/\lambda_j(M_h)$.

2. *From the note, we know that the eigenvalues of the continuous operator is* $j^2\pi^2$. *We also know that*

$$\lambda_j(M_h^{-1}A_h) = \frac{6}{h^2}\frac{1-\cos(\pi j h)}{2+\cos(\pi j h)} = j^2\pi^2 + \frac{1}{12}\pi^4 j^4 h^2 + \cdots.$$

So the approximation is of order 2.
For the second question:

$$\frac{6}{h^2}\frac{1-\cos(\pi j h)}{2+\cos(\pi j h)} > j^2\pi^2 \qquad \text{for } j = 1, 2, \cdots, N$$

since

$$6(1-\cos(x)) + x^2(2+\cos(x)) > 0 \qquad \text{for all } x \in (0,\pi).$$

3. *To solve the system exactly, you need N iterations. If you are happy with some approximation, use the error bound*

$$\|\mathbf{e}_k\|_{M_h} \leq 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k \|\mathbf{e}_0\|_{M_h}$$

where $\mathbf{e}_k = \mathbf{z}^k - \mathbf{z}^*$, $\|\mathbf{x}\|^2_{M_h} = \mathbf{x}^T M_h \mathbf{x}$, *and the condition number is* $\kappa = \lambda_{max}(M_h)/\lambda_{min}(M_h)$. *For the matrix in question,* $\kappa \approx 3$, *so* $(\sqrt{\kappa}-1)/(\sqrt{\kappa}+1) = K \approx 0.268$. *To be sure the error is reduced by a factor of* τ, *then*

$$2K^k < \tau \qquad \Rightarrow \qquad k > \frac{\log(\tau/2)}{\log(K)}.$$

To take some examples:

τ	10^{-2}	10^{-4}	10^{-6}	10^{-8}
k_{min}	5	8	12	15

4. *The eigenfunctions* $u_j(x)$ *are orthogonal on the inner product* $(\cdot,\cdot)$. *So*

$$v \in V_h \qquad \Rightarrow \qquad v = \sum_{j=1}^{N} \alpha_j u_j$$

and

$$a(v,v) = \sum_{j=1}^{N} \lambda_j \alpha_j^2 \|u_j\|_2^2 \geq \lambda_{min} \sum_{j=1}^{N} \alpha_j^2 \|u_j\|_2^2 = \lambda_{min}(v,v).$$

Similar arguments can be used to prove that $a(v,v) \leq \lambda_{max}(v,v)$.

5. *The finite difference approximation becomes*

$$\frac{1}{h^2}(u_{i-1,j} + 2u_{i,j} - u_{i+1,j}) = \lambda_j u_{i,j}, \qquad u_{0,j} = u_{N+1,j} = 0,$$

with $h = 1/(N+1)$. *Which is satisfied for*

$$u_{i,j} = \sin(\pi j i h), \qquad \lambda_j = \frac{1}{h^2}(1-\cos(\pi j h)) \approx \pi^2 j^2 - \frac{1}{12}\pi^4 j^4 h^2 + \cdots.$$

And $\frac{1}{h^2}(1-\cos(\pi j h)) < \pi^2 j^2$ *for* $j = 1,2,\cdots,N$ *since* $2(1-\cos(x)) - x^2 < 0$ *for all* $x \in (0,\pi)$.

6. *Let us start with (28): The element* i *of the eigenvector* $\mathbf{u}_{h,j}$ *corresponding to the eigenvalue* λ_j *is given as*

$$(u_{h,i})_j = \sin(\pi j(ih))$$

which satisfies $A_h \mathbf{u}_{h,j} = \lambda_j \mathbf{u}_{h,j}$, *or*

$$\frac{1}{h}\big(-(u_{h,i-1})_j + 2(u_{h,i})_j - (u_{h,i+1})_j\big) = \lambda_j (u_{h,i})_j, \tag{1}$$

which becomes

$$\frac{1}{h}(-\sin(\pi j(i-1)h + 2\sin(\pi j i h) - \sin(\pi j(i+1)h) = \lambda_j \sin(\pi i j h).$$

Using (see the footnote)

$$\sin(\pi j(i-1)h) + \sin(\pi j(i+1)h) = 2\sin(\pi i j h)\cos(\pi j h)$$

we prove that (1) is satisfied with $\lambda_j = 2(1-\cos(\pi j h))$ *for* $j = 1,2,\cdots,N$.