Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods
Fall 2012

Exercise set 6

1 The conjugate gradient (CG) algorithm for solving

$$A\mathbf{u} = \mathbf{b}$$

is given by

Compute $\mathbf{r}_0 = \mathbf{b} - A\mathbf{u}_0$.
for $k = 0, 1, 2, \cdots$ **do**
$\alpha_k = \dfrac{\mathbf{r}_k^T \mathbf{r}_k}{\mathbf{p}_k^T A \mathbf{p}_k}$
$\mathbf{u}_{k+1} = \mathbf{u}_k + \alpha_k \mathbf{p}_k$
$\mathbf{r}_{k+1} = \mathbf{r}_k - \alpha_k A \mathbf{p}_k$
$\beta_k = \dfrac{\mathbf{r}_{k+1}^T \mathbf{r}_{k+1}}{\mathbf{r}_k^T \mathbf{r}_k}$
$\mathbf{p}_{k+1} = \mathbf{r}_{k+1} + \beta_k \mathbf{p}_k$
end for

In the following, we will assume that A is symmetric, positive definite.

In the lecture, it was proved that $\mathbf{r}_i^T \mathbf{r}_j = 0$ and $\mathbf{p}_i A \mathbf{p}_j = 0$ if $i \neq j$.

a) Prove that this algorithm coincide with the algorithm on p. 166 in Quarteroni, with no preconditioner ($P = I$).

b) If A is symmetric positive definite, prove that so is A^{-1}. ThusA^{-1} can be used to define a norm: $\|\mathbf{y}\|_{A^{-1}} = \sqrt{\mathbf{y}^T A^{-1} \mathbf{y}}$ for all $\mathbf{y} \in \mathbb{R}^N$.

c) From the derivation of the CG algorithm, we know that at each iteration, the method minimizes the *error* in th A-norm over all elements in $K^m(A; \mathbf{r}_0)$. Prove that, at each iteration, CG minimizes the *residual* in the A^{-1} norm.

d) Assume that the initial residual can be expressed as a sum of m eigenvectors of A, with $m \leq N$. Show that CG converges in m iterations in this case.

2 Consider the diffusion-transport problem:

$$\begin{aligned} -\mu u_{xx} + b u_x &= f, \qquad in \quad \Omega = (0, L) \\ u(0) &= 0 \\ u(L) &= 1 \end{aligned}$$

where μ and b are given constants, and $\mu > 0$.

a) Find the weak formulation of the problem.

b) Set up the Galerkin approximation, using a space $V_h = \text{span}\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$.

c) Let $V_h = X_h^1$. Set up the elemental matrix A^K for this problem.

d) Assume a uniform grid ($h = L/M$ is constant). Set up the global linear system to be solved in this case.

e) Let $L = 20$, $\mu = 0.04$ and $b = 2$. How small do you have to make the stepsize h to avoid oscillations in the numerical solution? How many elements would you need?

f) Confirm the results of **e)** numerically (MATLAB file enclosed).

g) How can you avoid the oscillations, while still using a solution based on finite element methods. Again, confirm your results numerically.

h) Explain the idea of the Gummel-Scharfetter scheme (bottom half of page 285). In particular, explain why the use of this scheme will solve the diffusion-transport equation exactly if the source term $f = 0$.
Confirm your results numerically.

i) Assume that you would rather use a variable stepsize scheme. Describe a typical row in the global linear system of equations.