Exercise 6, solutions.

a) Multiply by a test function v, and use partial integration on the first term:

$$-\mu\int_0^L u_x v_x\,dx - \Big/_0^L u_x v + b\int_0^L u_x v\,dx = \int_0^L f v\,dx$$

For the integrals to exist, we require $u, v \in H^1(\Omega)$ and $f \in L^2(\Omega)$. Further, the second term is zero if $v(0) = v(L) = 0$, that is $v \in H_0^1(\Omega)$. So we get

Find $u \in H^1(\Omega)$, $u(0) = 0$, $u(L) = L$, s.t.

$$\mu\int_0^L u_x v_x\,dx + b\int_0^L u_x v\,dx = \int_0^L f\cdot v\,dx\,, \quad v \in H_0^1(\Omega).$$

To make the trial and test space the same, let

$$u = \mathring{u} + R_g, \quad \text{where } R_g(0) = 0,\ R_g(L) = 1$$

fore some chosen lifting function R_g.
The weak formulation becomes

Find $\mathring{u} \in H_0^1(\Omega)$ s.t. $a(u, v) = F(v)$ $\forall v \in H_0^1(\Omega)$.

with

$$a(u,v) = \mu\int_0^L u_x v_x\,dx + b\int_0^L u_x v\,dx$$

$$F(v) = \int_0^L f\cdot v\,dx - \mu\int_0^L R_{g,x} v_x\,dx - b\int_0^L R_{g,x}\, v\,dx.$$

Notice the lack of symmetry in this case, $a(u,v) \neq a(v,u)$.

b) The Galerkin approximation is given by:

Find $\mathring{u}_h \in V_h$ s.t. $a(\mathring{u}_h, \varphi_i) = F(\varphi_i)$, $i = 1, \cdots, N$.

Let $\mathring{u}_h = \sum_{j=1}^N u_j \varphi_j$. Using the bi-linearity of a,

this is

Find $\underline{u} \in \mathbb{R}^N$ s.t. $\sum_{j=1}^N a(\varphi_j, \varphi_i) = F(\varphi_i)$, $i = 1, \cdots, N$

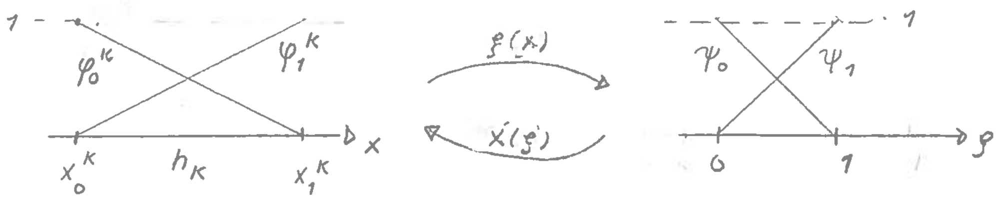
or simply

$$A_h \underline{u} = \underline{f},$$

Where
$$(A_h)_{ij} = \mu \int_0^L \frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial x} dx + b \int_0^L \frac{\partial \varphi_j}{\partial x} \cdot \varphi_i dx$$
$$f_i = \int_0^L f \cdot \varphi_i dx - \mu \int_0^L \frac{\partial R_g}{\partial x} \frac{\partial \varphi_i}{\partial x} dx - b \int_0^L \frac{\partial R_g}{\partial x} \varphi_i dx$$

And the solution is:
$$u_h = \overset{\circ}{u}_h + R_g = \sum_{j=1}^{N} u_j \varphi_j + R_g$$

c) Element K Reference element $\hat{K}$



$$\varphi_\alpha^K = \psi_\alpha(\xi(x))$$
$$\psi_0 = 1 - \xi$$
$$\psi_1 = \xi$$

Mapping $\hat{K} \Leftrightarrow K$: $x(\xi) = x_0^K + h_K \xi$
$$\xi(x) = \frac{x - x_0^K}{h_K}$$

So, element (α, β) of A^K, $\alpha, \beta = 0,1$ is
$$A^K_{\alpha,\beta} = \int_K a(\varphi_\alpha, \varphi_\beta) dx$$
$$= \mu \int_0^L \frac{\partial \varphi_\alpha}{\partial x} \frac{\partial \varphi_\beta}{\partial x} dx + b \int_0^L \frac{\partial \varphi_\beta}{\partial x} \varphi_\alpha dx$$
$$= \frac{\mu}{h_K} \int_0^1 \frac{\partial \psi_\alpha}{\partial \xi} \frac{\partial \psi_\beta}{\partial \xi} d\xi + b \int_0^1 \frac{\partial \psi_\beta}{\partial \xi} \psi_\alpha d\xi$$

$$A^K_{00} = \frac{\mu}{h_K} \int_0^1 1 \cdot 1 d\xi + b \int_0^1 (-1) \cdot (1 - \xi) d\xi = \frac{\mu}{h_K} - \frac{b}{2}$$
$$A^K_{01} = \frac{\mu}{h_K} \int_0^1 1 \cdot (-1) d\xi + b \int_0^1 1 \cdot (1 - \xi) d\xi = -\frac{\mu}{h_K} + \frac{b}{2}$$
$$A^K_{10} = \frac{\mu}{h_K} \int_0^1 (-1) 1 d\xi + b \int_0^1 (-1) \xi d\xi = -\frac{\mu}{h_K} - \frac{b}{2}$$
$$A^K_{11} = \frac{\mu}{h_K} \int_0^1 (-1)(-1) d\xi + b \int_0^1 1 \cdot \xi d\xi = \frac{\mu}{h_K} + \frac{b}{2}$$

So, the element matrix becomes:

$$A^{K} = \frac{\mu}{h_K}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{b}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

d) First, we need the mapping from the local to the global grid, wich in this case simply is

$$i = \theta(\alpha, k) = k - 1 + \alpha, \qquad k = 1, 2, \cdots, N-1, \qquad \alpha = 0, 1$$

such that, for each $x_i = ih$, we have a basis function satisfying

$$\varphi_i(x_j) = \delta_{ij}, \qquad \text{and} \qquad \varphi_i\big|_K = \varphi_{\theta(\alpha,k)}$$

The extended matrix $\tilde{A}_h$, which includes the bounday functions, becomes

$$\tilde{A} = \frac{\mu}{h}\begin{bmatrix} 1 & -1 & & & \\ -1 & (1+1) & -1 & & \\ & -1 & (1+1) & -1 & \\ & & -1 & \ddots & \ddots \\ & & & \ddots & (1+1) & -1 \\ & & & & -1 & 1 \end{bmatrix}$$

$$+ \frac{b}{2}\begin{bmatrix} -1 & 1 & & & \\ -1 & (1-1) & 1 & & \\ & -1 & (1-1) & 1 & \\ & & & \ddots & \\ & & & & (1-1) & 1 \\ & & & & -1 & 1 \end{bmatrix}$$

$$= \frac{\mu}{h}\begin{bmatrix} 1 & 1 & & 0 \\ 1 & 2 & \ddots & \\ & \ddots & \ddots & \\ 0 & & 2 & 1 \\ & & 1 & 1 \end{bmatrix} + \frac{b}{2}\begin{bmatrix} -1 & 1 & & 0 \\ -1 & 0 & \ddots & \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ & & -1 & 1 \end{bmatrix}$$

In the derivation of the weak formulation, we could choose the lifting function $R_g \in H^1(\Omega)$, as long as it satisfies the boundary conditions. Now we choose

$$R_g = u(0)\varphi_0 + u(L)\varphi_N, \quad \text{such that}$$

and $u_h = \sum_{i=0}^{N} u_i \varphi_i$, with $u_0 = u(0)$ and $u_N = u(L)$,

The numerical solution satisfies

$$\tilde{A}\underline{\tilde{u}} = \underline{\tilde{f}} \qquad , \qquad \tilde{u} = \begin{bmatrix} u_0 \\ \underline{u} \\ u_N \end{bmatrix}$$

and $f_i = \int_0^L f\varphi_i \, dx, \quad i = 0, \cdots, M.$

Since $u_0 = 0$ and $u_M = 1$, we can remove the first and last row of the system, and move the first and last column to the rhs. Thus the global system is:

$$(\mu A_d + b A_t)\underline{u} = \underline{f} + \underline{b}$$

where

$$A_d = \text{tridiag}\{-1, 2, -1\} \cdot 1/h$$

$$A_t = \text{tridiag}\{-1, 0, 1\} \cdot 1/2$$

$$\underline{b} = (0, \cdots, 0, \ \mu/h - b/2)^T$$

$$\underline{f} = (f_1, \cdots, f_{M-1})^T.$$

e) Our scheme becomes, using $f = 0$, becomes

$$\frac{\mu}{h}(-u_{i-1} + 2u_i - u_{i+1}) + \frac{b}{2}(-u_{i-1} + u_{i+1}) = 0$$

Try the solutions $u_i = r^i$. Inserting this into the formula above gives

$$\left(-\frac{\mu}{h} + \frac{b}{2}\right)r^2 - \frac{2\mu}{h} r + \left(-\frac{\mu}{h} - \frac{b}{2}\right) = 0$$

Which has solutions

$$r_1 = 1, \qquad r_2 = \frac{1 + \frac{bh}{2\mu}}{1 - \frac{bh}{2\mu}}$$

and the total Solution is

$$u_i = C_1 r_1^i + C_2 r_2^i$$

the constants C_1 and C_2 are given from the boundary conditions. It is clear that oscillations will take place if $r_2 < 0$, that is, when

$$\left|\frac{bh}{2\mu}\right| > 1 \qquad \text{or} \qquad h > \frac{2\mu}{|b|}$$

Which in our case means

$$h > \frac{2 \cdot 0.04}{2} = 0.04, \qquad \text{or } M < \frac{L}{0.04} = 500.$$

f) Use artifical diffusion: Solve the modified problem

$$-\mu(1 + \varphi_h)u_{xx} + bu_x = f$$

s.t.

$$\frac{|b|h}{2\mu(1+\varphi_h)} < 1, \qquad \text{f.ex.} \quad \varphi_h = \frac{|b|h}{2\mu}$$

Which will lead to the standard upwind scheme.

b) You are not really expected to solve this unless you participated at the lecture 17.11.
In this note, I will use an alternative approach.

Given

$$-\mu u_{xx} + b u_x = f$$

which can be rewritten as

$$-\mu \psi_x = f \quad , \quad \text{with} \quad \psi = u_x - \beta u, \quad \beta = b/\mu .$$

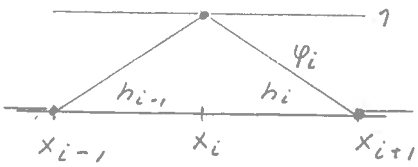
with some appropriate boundary conditions, (Dirichlet)
The weak formulation for this is

$$-\mu \int_\Omega \psi_x v \, dx = \int_\Omega \psi v_x \, dx = \int_\Omega f \cdot v \, dx \quad , \quad \forall v \in H_0^1(\Omega)$$

and the Galerkin approximation becomes

$$(*) \qquad +\mu \int_\Omega \psi_h \frac{\partial \varphi_i}{\partial x} \, dx = \int_\Omega f \cdot \varphi_i \, dx$$

Let $V_h = X_0^1(\Omega)$, or



Then, on each interval (x_i, x_{i+1}), ψ_h is linear, and on each interval.

Next, approximate $\psi_h = \psi_{i+1/2}$ (const.) on (x_i, x_{i+1})
and solve

$$v_x - \beta v = \psi_{i+1/2} \quad , \quad v(x_i) = u_i \; , \; v(x_{i+1}) = u_{i+1}$$

which gives

$$\psi_{i+1/2} = \beta \frac{e^{\beta h} u_i - u_{i+1}}{e^{\beta h} - 1}$$

Inserted into $(*)$ this becomes

$$\frac{\mu \beta}{h} \cdot \frac{1}{e^{\beta h} - 1} \left(e^{\beta h} u_{i-1} - u_i - e^{\beta h} u_i + u_{i+1} \right) = \int_{x_{i-1}}^{x_{i+1}} f \cdot \varphi_i \, dx$$

for $i = 1, 2, \cdots, M-1$.

Since u_0 and u_M are known, this is diagonal dominant system of equations, with a unique solution.

And it can be rewritten into the form

$$\frac{\mu}{h}(1 + \Phi_h)(-u_{i-1} + 2u_i - u_{i+1}) + \frac{1}{2}(-u_{i-1} + u_{i+1}) = \int f \cdot \varphi_i \, dx$$

with

$$\phi_h = -1 + \frac{\beta h}{2} + \frac{\beta h}{e^{\beta h} - 1} .$$

It is pretty clear that this is the exact solution for $f = 0$.

i) If $h_k \neq h_{k+1}$, then a typical row is

$$\mu \left(\frac{1}{h_k} (-u_{k-1} + u_k) + \frac{1}{h_{k+1}} (+u_k - u_{k+1}) \right)$$
$$+ \frac{b}{2} \left(\frac{1}{h_k} (-u_{k-1} + u_k) + \frac{1}{h_{k+1}} (-u_k + u_{k+1}) \right) = \int_{x_{k-1}}^{x_{k+1}} f \varphi_k \, dx$$