Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods
Fall 2013

Project 1

1 Consider the problem

$$-u_{xx} = 1, \quad 0 < x < 1, \quad u(0) = u(1) = 0.$$

a) Derive the exact solution u.

b) Show by explicit calculations that

$$\int_0^1 u_x v_x dx = \int_0^1 v dx$$

for all sufficiently smooth v satisfying $v(0) = v(1) = 0$.

c) Compute $J(u)$, where

$$J(v) = \frac{1}{2}\int_0^1 v_x^2 dx - \int_0^1 v dx$$

d) Let $w_1(x) = a_1 \sin(\pi x)$. Find the value of the amplitude a_1 which minimizes $J(w_1)$. How does a_1 compare with the maximum of the exact solution u?

e) Show that $J(w_1) > J(u)$. Is there a big difference?

f) Let $\varphi_i = \sin\left((2i-1)\pi x\right)$, $i = 1, 2, 3, \ldots$. These functions are infinitely differentiable, and they all satisfy $\varphi_i(0) = \varphi_i(1) = 0$. Compute

$$a_{ij} = \int_0^1 \varphi_j' \varphi_i' dx \quad \text{and} \quad b_i = \int_0^1 \varphi_i dx.$$

g) Let $V_N = \text{span}\,\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$. Set up and solve the problem

$$\text{Find } w_N \in V_N \text{ such that } \int_0^1 w_{N,x} v_x dx = \int_0^1 v dx, \quad \forall v \in V_N.$$

h) Plot the error $u - w_N$ for $N = 1, 2, 3$.

2 Given the weak statement:

$$\text{Find } u \in V \quad \text{s.t.} \quad a(u,v) = F(v), \qquad \forall v \in V \tag{1}$$

and the minimization principle:

$$u = \arg\min_{u \in V} J(v), \qquad \text{with} \quad J(v) = \frac{1}{2}a(v,v) - F(v). \tag{2}$$

a) Show that (1) and (2) are equivalent whenever a is bilinear, symmetric and positive definite, and F is linear. State clearly which properties you are using in you arguments.

b) Take $V = \mathbb{R}^n$, and just show, by appropriate choice of a and F, that the minimizer $u \in V$ of $J(v) = \frac{1}{2}v^T G v - v^T b$ for any symmetric, positive definite matrix $G \in \mathbb{R}^{n \times n}$ and $v \in \mathbb{R}^n$ satisfies $Gu = b$.

3 Write a code for solving the equation

$$-u_{xx} = x^4, \quad 0 < x < 1, \quad u(0) = 0, \quad u(1) = 0.$$

using the finite element method with equidistant grid ($x_i = ih,\ h = 1/N$), and the basis functions

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h}, & \text{for } x_{i-1} \le x \le x_i, \\ \frac{x_{i+1} - x}{h} & \text{for } x_i \le x \le x_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

for $i = 0, 1, 2, \ldots, N$. Compare the numerical solution with the exact solution, and plot the error.