Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods
Fall 2013

Project 2

1 If you are not familiar with the the Lebesgue spaces $L^p(\Omega)$ and the Sobolov spaces $H^p(\Omega)$, you should read section 2.3.1 and 2.4.0-2.4.2.

True or false:

a) The set $S = \left\{v \in C^0(0,1)\ :\ v\left(\frac{1}{2}\right) = 1\right\}$ is a linear (vector) space.

b) For $X = H_0^1\left((0,1)\right),\ L(v) = \int_0^1 xv dx$ is a linear functional.

c) For $Z = \mathbb{R}$, $(x,y)_Z = |x|\,|y|$ is a valid inner product.

d) The only v in $H^1(\Omega)$ for which $|w|_{H^1(\Omega)}$ (the H^1 semi-norm) is zero is $v = 0$.

e) The function $v = x^{3/4}$ is in $L^2\left((0,1)\right)$; in $H^1((0,1))$; in $H^2\left((0,1)\right)$.

f) For $v = e^{-10x}$, $|v|_{H^2((0,1))} = |v|_{H^1((0,1))}$.

2 Consider the fourth-order problem:

$$u_{xxxx} = f \quad \text{in } \Omega = (0,1)\,,$$
$$u(0) = u_x(0) = u(1) = u_x(1) = 0.$$

This "biharmonic" equation is relevant to, amongst other applications, the bending of beams.

a) Find a symmetric, positive form a over V and a linear form F such that the solution u of the equation satisfies

$$a(u,v) = F(v),\ \forall v \in V.$$

b) How should V be defined?

c) Do you think that $F(v) = v_x\left(\frac{1}{2}\right)$ is a linear, bounded functional on V?

3 Consider the problem with a discontinuous jump in conductivities:

$$-\kappa^L u^L_{xx} = f^L, \qquad 0 < x < \frac{1}{2},$$
$$-\kappa^R u^R_{xx} = f^R, \qquad \frac{1}{2} < x < 1,$$

with boundary conditions

$$\begin{aligned} u^L(0) = 0, \qquad & u^R(1) = 0, \\ u^L\left(\tfrac{1}{2}\right) &= u^R\left(\tfrac{1}{2}\right), \\ \kappa^L u_x^L\left(\tfrac{1}{2}\right) &= \kappa^R u_x^R\left(\tfrac{1}{2}\right), \qquad \text{(continuity of flux)}. \end{aligned}$$

Here, κ^L and κ^R are strictly positive. Let $V = \left\{ v \in H^1((0,1)) \,:\, v(0) = v(1) = 0 \right\}$. Find a and F such that the solution u satisfies

$$a(u, v) = F(v), \qquad \forall v \in V.$$

4 Given the Helmholtz problem

$$\begin{aligned} -u_{xx} + \sigma u = f \text{ on } (0,1), \\ u(0) = u(1) = 0. \end{aligned}$$

where $\sigma > 0$ is a constant. Set up the weak form for this problem. Show that, when this problem is solved by a Galerkin method, using $V_h = \text{span}\,\{\phi_i\}_{i=1}^{N}$, the discrete problem can be written as

$$(A + \sigma M)\mathbf{u} = \mathbf{f}.$$

Set up the matrix M for $V_h = X_h^1$ on a uniform grid.