2.48. Sketch the Mohr's circles and determine the maximum shear stress for each of the following stress states:

$$(a)\quad \sigma_{ij} = \begin{pmatrix} \tau & \tau & 0 \\ \tau & \tau & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (b)\quad \sigma_{ij} = \begin{pmatrix} \tau & 0 & 0 \\ 0 & -\tau & 0 \\ 0 & 0 & -2\tau \end{pmatrix}$$

Ans. (a) $\sigma_S = \tau$, (b) $\sigma_S = 3\tau/2$

2.49. Use the result given in Problem 1.58, page 39, together with the stress transformation law (*2.27*), page 50, to show that $\epsilon_{ijk}\epsilon_{pqm}\sigma_{ip}\sigma_{jq}\sigma_{km}$ is an invariant.

2.50. In a continuum, the stress field is given by the tensor

$$\sigma_{ij} = \begin{pmatrix} x_1^2 x_2 & (1-x_2^2)x_1 & 0 \\ (1-x_2^2)x_1 & (x_2^3 - 3x_2)/3 & 0 \\ 0 & 0 & 2x_3^2 \end{pmatrix}$$

Determine (*a*) the body force distribution if the equilibrium equations are to be satisfied throughout the field, (*b*) the principal stress values at the point $P(a, 0, 2\sqrt{a})$, (*c*) the maximum shear stress at P, (*d*) the principal deviator stresses at P.

Ans. (a) $b_3 = -4x_3$, (b) $a, -a, 8a$, (c) $\pm 4.5a$, (d) $-11a/3, -5a/3, 16a/3$

Chapter 3

Deformation and Strain

3.1 PARTICLES AND POINTS

In the kinematics of continua, the meaning of the word "point" must be clearly understood since it may be construed to refer either to a "point" in space, or to a "point" of a continuum. To avoid misunderstanding, the term "point" will be used exclusively to designate a location in fixed space. The word "particle" will denote a small volumetric element, or "material point", of a continuum. In brief, a *point* is a place in space, a *particle* is a small part of a material continuum.

3.2 CONTINUUM CONFIGURATION. DEFORMATION AND FLOW CONCEPTS

At any instant of time t, a continuum having a volume V and bounding surface S will occupy a certain region R of physical space. The identification of the particles of the continuum with the points of the space it occupies at time t by reference to a suitable set of coordinate axes is said to specify the *configuration* of the continuum at that instant.

The term *deformation* refers to a change in the shape of the continuum between some initial (undeformed) configuration and a subsequent (deformed) configuration. The emphasis in deformation studies is on the initial and final configurations. No attention is given to intermediate configurations or to the particular sequence of configurations by which the deformation occurs. By contrast, the word *flow* is used to designate the continuing state of motion of a continuum. Indeed, a configuration history is inherent in flow investigations for which the specification of a time-dependent velocity field is given.

3.3 POSITION VECTOR. DISPLACEMENT VECTOR

In Fig. 3-1 the undeformed configuration of a material continuum at time $t = 0$ is shown together with the deformed configuration of the same continuum at a later time $t = t$. For the present development it is useful to refer the initial and final configurations to separate coordinate axes as in the figure.

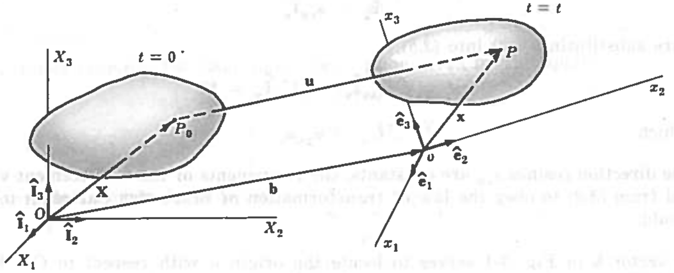

Fig. 3-1

Accordingly, in the initial configuration a representative particle of the continuum occupies a point P_0 in space and has the *position vector*

$$\mathbf{X} = X_1\hat{\mathbf{I}}_1 + X_2\hat{\mathbf{I}}_2 + X_3\hat{\mathbf{I}}_3 = X_K\hat{\mathbf{I}}_K \tag{3.1}$$

with respect to the rectangular Cartesian axes $OX_1X_2X_3$. Upper-case letters are used as indices in (*3.1*) and will appear as such in several equations that follow, but their use as summation indices is restricted to this section. In the remainder of the book upper-case subscripts or superscripts serve as labels only. Their use here is to emphasize the connection of certain expressions with the coordinates $(X_1X_2X_3)$, which are called the *material coordinates*. In the deformed configuration the particle originally at P_0 is located at the point P and has the position vector

$$\mathbf{x} = x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3 = x_i\hat{\mathbf{e}}_i \tag{3.2}$$

when referred to the rectangular Cartesian axes $ox_1x_2x_3$. Here lower-case letters are used as subscripts to identify with the coordinates $(x_1x_2x_3)$ which give the current position of the particle and are frequently called the *spatial coordinates*.

The relative orientation of the material axes $OX_1X_2X_3$ and the spatial axes $ox_1x_2x_3$ is specified through direction cosines α_{kK} and α_{Kk}, which are defined by the dot products of unit vectors as

$$\hat{\mathbf{e}}_k \cdot \hat{\mathbf{I}}_K = \hat{\mathbf{I}}_K \cdot \hat{\mathbf{e}}_k = \alpha_{kK} = \alpha_{Kk} \tag{3.3}$$

No summation is implied by the indices in these expressions since k and K are distinct indices. Inasmuch as Kronecker deltas are designated by the equations $\hat{\mathbf{I}}_K \cdot \hat{\mathbf{I}}_P = \delta_{KP}$ and $\hat{\mathbf{e}}_k \cdot \hat{\mathbf{e}}_p = \delta_{kp}$, the *orthogonality conditions* between spatial and material axes take the form

$$\alpha_{kK}\alpha_{Kp} = \alpha_{kK}\alpha_{pK} = \delta_{kp}; \quad \alpha_{Kp}\alpha_{Mp} = \alpha_{pK}\alpha_{pM} = \delta_{KM} \tag{3.4}$$

In Fig. 3-1 the vector $\mathbf{u}$ joining the points P_0 and P (the initial and final positions, respectively, of the particle), is known as the *displacement vector*. This vector may be expressed as

$$\mathbf{u} = u_k\hat{\mathbf{e}}_k \tag{3.5}$$

or alternatively as

$$\mathbf{U} = U_K\hat{\mathbf{I}}_K \tag{3.6}$$

in which the components U_K and u_k are interrelated through the direction cosines α_{kK}. From (*1.89*) the unit vector $\hat{\mathbf{e}}_k$ is expressed in terms of the material base vectors $\hat{\mathbf{I}}_K$ as

$$\hat{\mathbf{e}}_k = \alpha_{kK}\hat{\mathbf{I}}_K \tag{3.7}$$

Therefore substituting (*3.7*) into (*3.5*),

$$\mathbf{u} = u_k(\alpha_{kK}\hat{\mathbf{I}}_K) = U_K\hat{\mathbf{I}}_K = \mathbf{U} \tag{3.8}$$

from which

$$U_K = \alpha_{kK}u_k \tag{3.9}$$

Since the direction cosines α_{kK} are constants, the components of the displacement vector are observed from (*3.9*) to obey the law of transformation of first-order Cartesian tensors, as they should.

The vector $\mathbf{b}$ in Fig. 3-1 serves to locate the origin o with respect to O. From the geometry of the figure,

$$\mathbf{u} = \mathbf{b} + \mathbf{x} - \mathbf{X} \tag{3.10}$$

Very often in continuum mechanics it is possible to consider the coordinate systems $OX_1X_2X_3$ and $ox_1x_2x_3$ superimposed, with $\mathbf{b} \equiv 0$, so that (*3.10*) becomes

$$\mathbf{u} = \mathbf{x} - \mathbf{X} \tag{3.11}$$

In Cartesian component form this equation is given by the general expression

$$u_k = x_k - \alpha_{kK}X_K \tag{3.12}$$

However, for superimposed axes the unit triads of base vectors for the two systems are identical, which results in the direction cosine symbols α_{kK} becoming Kronecker deltas. Accordingly, (*3.12*) reduces to

$$u_k = x_k - X_k \tag{3.13}$$

in which only lower-case subscripts appear. In the remainder of this book, unless specifically stated otherwise, the material and spatial axes are assumed *superimposed* and hence only lower-case indices will be used.

3.4 LAGRANGIAN AND EULERIAN DESCRIPTIONS

When a continuum undergoes deformation (or flow), the particles of the continuum move along various paths in space. This motion may be expressed by equations of the form

$$x_i = x_i(X_1, X_2, X_3, t) = x_i(\mathbf{X}, t) \quad \text{or} \quad \mathbf{x} = \mathbf{x}(\mathbf{X}, t) \tag{3.14}$$

which give the present location x_i of the particle that occupied the point $(X_1X_2X_3)$ at time $t = 0$. Also, (*3.14*) may be interpreted as a mapping of the initial configuration into the current configuration. It is assumed that such a mapping is one-to-one and continuous, with continuous partial derivatives to whatever order is required. The description of motion or deformation expressed by (*3.14*) is known as the *Lagrangian* formulation.

If, on the other hand, the motion or deformation is given through equations of the form

$$X_i = X_i(x_1, x_2, x_3, t) = X_i(\mathbf{x}, t) \quad \text{or} \quad \mathbf{X} = \mathbf{X}(\mathbf{x}, t) \tag{3.15}$$

in which the independent variables are the coordinates x_i and t, the description is known as the *Eulerian* formulation. This description may be viewed as one which provides a tracing to its original position of the particle that now occupies the location (x_1, x_2, x_3). If (*3.15*) is a continuous one-to-one mapping with continuous partial derivatives, as was also assumed for (*3.14*), the two mappings are the unique inverses of one another. A necessary and sufficient condition for the inverse functions to exist is that the Jacobian

$$J = \left|\frac{\partial x_i}{\partial X_j}\right| \tag{3.16}$$

should not vanish.

As a simple example, the Lagrangian description given by the equations

$$\begin{aligned} x_1 &= X_1 + X_2(e^t - 1) \\ x_2 &= X_1(e^{-t} - 1) + X_2 \\ x_3 &= X_3 \end{aligned} \tag{3.17}$$

has the inverse Eulerian formulation,

$$\begin{aligned} X_1 &= \frac{-x_1 + x_2(e^t - 1)}{1 - e^t - e^{-t}} \\ X_2 &= \frac{x_1(e^{-t} - 1) - x_2}{1 - e^t - e^{-t}} \\ X_3 &= x_3 \end{aligned} \tag{3.18}$$

3.5 DEFORMATION GRADIENTS. DISPLACEMENT GRADIENTS

Partial differentiation of (*3.14*) with respect to X_j produces the tensor $\partial x_i/\partial X_j$ which is called the *material deformation gradient*. In symbolic notation, $\partial x_i/\partial X_j$ is represented by the dyadic

$$\mathbf{F} = \mathbf{x}\nabla_{\mathbf{X}} \equiv \frac{\partial \mathbf{x}}{\partial X_1}\hat{\mathbf{e}}_1 + \frac{\partial \mathbf{x}}{\partial X_2}\hat{\mathbf{e}}_2 + \frac{\partial \mathbf{x}}{\partial X_3}\hat{\mathbf{e}}_3 \tag{3.19}$$

in which the differential operator $\nabla_{\mathbf{X}} = \frac{\partial}{\partial X_i}\hat{\mathbf{e}}_i$ is applied from the right (as shown explicitly in the equation). The matrix form of $\mathbf{F}$ serves to further clarify this property of the operator $\nabla_{\mathbf{X}}$ when it appears as the consequent of a dyad. Thus

$$\mathcal{F} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\left[\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \frac{\partial}{\partial X_3}\right] = \begin{bmatrix} \partial x_1/\partial X_1 & \partial x_1/\partial X_2 & \partial x_1/\partial X_3 \\ \partial x_2/\partial X_1 & \partial x_2/\partial X_2 & \partial x_2/\partial X_3 \\ \partial x_3/\partial X_1 & \partial x_3/\partial X_2 & \partial x_3/\partial X_3 \end{bmatrix} = [\partial x_i/\partial X_j] \tag{3.20}$$

Partial differentiation of (*3.15*) with respect to x_j produces the tensor $\partial X_i/\partial x_j$ which is called the *spatial deformation gradient*. This tensor is represented by the dyadic

$$\mathbf{H} = \mathbf{X}\nabla_{\mathbf{x}} \equiv \frac{\partial \mathbf{X}}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{\partial \mathbf{X}}{\partial x_2}\hat{\mathbf{e}}_2 + \frac{\partial \mathbf{X}}{\partial x_3}\hat{\mathbf{e}}_3 \tag{3.21}$$

having a matrix form

$$\mathcal{H} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right] = \begin{bmatrix} \partial X_1/\partial x_1 & \partial X_1/\partial x_2 & \partial X_1/\partial x_3 \\ \partial X_2/\partial x_1 & \partial X_2/\partial x_2 & \partial X_2/\partial x_3 \\ \partial X_3/\partial x_1 & \partial X_3/\partial x_2 & \partial X_3/\partial x_3 \end{bmatrix} = [\partial X_i/\partial x_j] \tag{3.22}$$

The material and spatial deformation tensors are interrelated through the well-known chain rule for partial differentiation,

$$\frac{\partial x_i}{\partial X_j}\frac{\partial X_j}{\partial x_k} = \frac{\partial X_i}{\partial x_j}\frac{\partial x_j}{\partial X_k} = \delta_{ik} \tag{3.23}$$

Partial differentiation of the displacement vector u_i with respect to the coordinates produces either the *material displacement gradient* $\partial u_i/\partial X_j$, or the *spatial displacement gradient* $\partial u_i/\partial x_j$. From (*3.13*), which expresses u_i as a difference of coordinates, these tensors are given in terms of the deformation gradients as the material gradient

$$\frac{\partial u_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j} - \delta_{ij} \quad \text{or} \quad \mathbf{J} \equiv \mathbf{u}\nabla_{\mathbf{X}} = \mathbf{F} - \mathbf{I} \tag{3.24}$$

and the spatial gradient

$$\frac{\partial u_i}{\partial x_j} = \delta_{ij} - \frac{\partial X_i}{\partial x_j} \quad \text{or} \quad \mathbf{K} \equiv \mathbf{u}\nabla_{\mathbf{x}} = \mathbf{I} - \mathbf{H} \tag{3.25}$$

In the usual manner, the matrix forms of $\mathbf{J}$ and $\mathbf{K}$ are respectively

$$\mathcal{J} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\left[\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \frac{\partial}{\partial X_3}\right] = \begin{bmatrix} \partial u_1/\partial X_1 & \partial u_1/\partial X_2 & \partial u_1/\partial X_3 \\ \partial u_2/\partial X_1 & \partial u_2/\partial X_2 & \partial u_2/\partial X_3 \\ \partial u_3/\partial X_1 & \partial u_3/\partial X_2 & \partial u_3/\partial X_3 \end{bmatrix} = [\partial u_i/\partial X_j] \tag{3.26}$$

and

$$\mathcal{K} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right] = \begin{bmatrix} \partial u_1/\partial x_1 & \partial u_1/\partial x_2 & \partial u_1/\partial x_3 \\ \partial u_2/\partial x_1 & \partial u_2/\partial x_2 & \partial u_2/\partial x_3 \\ \partial u_3/\partial x_1 & \partial u_3/\partial x_2 & \partial u_3/\partial x_3 \end{bmatrix} = [\partial u_i/\partial x_j] \tag{3.27}$$

3.6 DEFORMATION TENSORS. FINITE STRAIN TENSORS

In Fig. 3-2 the initial (undeformed) and final (deformed) configurations of a continuum are referred to the superposed rectangular Cartesian coordinate axes $OX_1X_2X_3$ and $ox_1x_2x_3$. The neighboring particles which occupy points P_0 and Q_0 before deformation, move to points P and Q respectively in the deformed configuration.

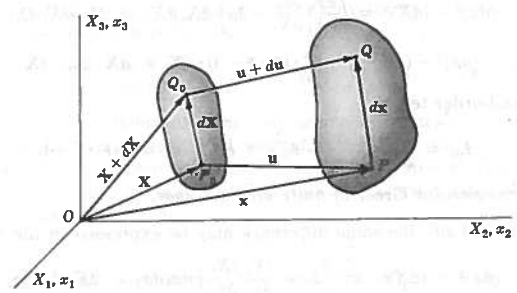

Fig. 3-2

The square of the differential element of length between P_0 and Q_0 is

$$(dX)^2 = d\mathbf{X}\cdot d\mathbf{X} = dX_i\,dX_i = \delta_{ij}\,dX_i\,dX_j \tag{3.28}$$

From (*3.15*), the distance differential dX_i is seen to be

$$dX_i = \frac{\partial X_i}{\partial x_j}dx_j \quad \text{or} \quad d\mathbf{X} = \mathbf{H}\cdot d\mathbf{x} \tag{3.29}$$

so that the squared length $(dX)^2$ in (*3.28*) may be written

$$(dX)^2 = \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}dx_i\,dx_j = C_{ij}\,dx_i\,dx_j \quad \text{or} \quad (dX)^2 = d\mathbf{x}\cdot\mathbf{C}\cdot d\mathbf{x} \tag{3.30}$$

in which the second-order tensor

$$C_{ij} = \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j} \quad \text{or} \quad \mathbf{C} = \mathbf{H}_c\cdot\mathbf{H} \tag{3.31}$$

is known as *Cauchy's deformation tensor*.

In the deformed configuration, the square of the differential element of length between P and Q is

$$(dx)^2 = d\mathbf{x}\cdot d\mathbf{x} = dx_i\,dx_i = \delta_{ij}\,dx_i\,dx_j \tag{3.32}$$

From (*3.14*) the distance differential here is

$$dx_i = \frac{\partial x_i}{\partial X_j}dX_j \quad \text{or} \quad d\mathbf{x} = \mathbf{F}\cdot d\mathbf{X} \tag{3.33}$$

so that the squared length $(dx)^2$ in (*3.32*) may be written

$$(dx)^2 = \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j}dX_i\,dX_j = G_{ij}\,dX_i\,dX_j \quad \text{or} \quad (dx)^2 = d\mathbf{X}\cdot\mathbf{G}\cdot d\mathbf{X} \tag{3.34}$$

in which the second-order tensor

$$G_{ij} = \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} \quad \text{or} \quad \mathbf{G} = \mathbf{F}_c\cdot\mathbf{F} \tag{3.35}$$

is known as *Green's deformation tensor*.

The difference $(dx)^2-(dX)^2$ for two neighboring particles of a continuum is used as the *measure of deformation* that occurs in the neighborhood of the particles between the initial and final configurations. If this difference is identically zero for all neighboring particles of a continuum, a *rigid displacement* is said to occur. Using (3.34) and (3.28), this difference may be expressed in the form

$$(dx)^2 - (dX)^2 = \left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right) dX_i\, dX_j = 2L_{ij}\, dX_i\, dX_j$$

or
$$(dx)^2 - (dX)^2 = d\mathbf{X}\cdot(\mathbf{F}_c\cdot\mathbf{F} - \mathbf{I})\cdot d\mathbf{X} = d\mathbf{X}\cdot 2\mathbf{L}_G\cdot d\mathbf{X} \qquad (3.36)$$

in which the second-order tensor

$$L_{ij} = \frac{1}{2}\left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right) \quad \text{or} \quad \mathbf{L}_G = \tfrac{1}{2}(\mathbf{F}_c\cdot\mathbf{F} - \mathbf{I}) \qquad (3.37)$$

is called the *Lagrangian* (or Green's) *finite strain tensor*.

Using (3.32) and (3.30), the same difference may be expressed in the form

$$(dx)^2 - (dX)^2 = \left(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right) dx_i\, dx_j = 2E_{ij}\, dx_i\, dx_j$$

or
$$(dx)^2 - (dX)^2 = d\mathbf{x}\cdot(\mathbf{I} - \mathbf{H}_c\cdot\mathbf{H})\cdot d\mathbf{x} = d\mathbf{x}\cdot 2\mathbf{E}_A\cdot d\mathbf{x} \qquad (3.38)$$

in which the second-order tensor

$$E_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right) \quad \text{or} \quad \mathbf{E}_A = \tfrac{1}{2}(\mathbf{I} - \mathbf{H}_c\cdot\mathbf{H}) \qquad (3.39)$$

is called the *Eulerian* (or Almansi's) *finite strain tensor*.

An especially useful form of the Lagrangian and Eulerian finite strain tensors is that in which these tensors appear as functions of the displacement gradients. Thus if $\partial x_i/\partial X_j$ from (3.24) is substituted into (3.37), the result after some simple algebraic manipulations is the Lagrangian finite strain tensor in the form

$$L_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right) \quad \text{or} \quad \mathbf{L}_G = \tfrac{1}{2}(\mathbf{J} + \mathbf{J}_c + \mathbf{J}_c\cdot\mathbf{J}) \qquad (3.40)$$

In the same manner, if $\partial X_i/\partial x_j$ from (3.25) is substituted into (3.39), the result is the Eulerian finite strain tensor in the form

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) \quad \text{or} \quad \mathbf{E}_A = \tfrac{1}{2}(\mathbf{K} + \mathbf{K}_c - \mathbf{K}_c\cdot\mathbf{K}) \qquad (3.41)$$

The matrix representations of (3.40) and (3.41) may be written directly from (3.26) and (3.27) respectively.

3.7 SMALL DEFORMATION THEORY. INFINITESIMAL STRAIN TENSORS

The so-called *small deformation theory* of continuum mechanics has as its basic condition the requirement that the displacement gradients be small compared to unity. The fundamental measure of deformation is the difference $(dx)^2-(dX)^2$, which may be expressed in terms of the displacement gradients by inserting (3.40) and (3.41) into (3.36) and (3.38) respectively. If the displacement gradients are small, the finite strain tensors in (3.36) and (3.38) reduce to infinitesimal strain tensors, and the resulting equations represent small deformations.

In (3.40), if the displacement gradient components $\partial u_i/\partial X_j$ are each small compared to unity, the product terms are negligible and may be dropped. The resulting tensor is the *Lagrangian infinitesimal strain tensor*, which is denoted by

$$l_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \quad \text{or} \quad \mathbf{L} = \tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{X}} + \nabla_{\mathbf{X}}\mathbf{u}) = \tfrac{1}{2}(\mathbf{J} + \mathbf{J}_c) \qquad (3.42)$$

Likewise for $\partial u_i/\partial x_j \ll 1$ in (3.41), the product terms may be dropped to yield the *Eulerian infinitesimal strain tensor*, which is denoted by

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{or} \quad \mathbf{E} = \tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{x}} + \nabla_{\mathbf{x}}\mathbf{u}) = \tfrac{1}{2}(\mathbf{K} + \mathbf{K}_c) \qquad (3.43)$$

If both the displacement gradients and the displacements themselves are small, there is very little difference in the material and spatial coordinates of a continuum particle. Accordingly the material gradient components $\partial u_i/\partial X_j$ and spatial gradient components $\partial u_i/\partial x_j$ are very nearly equal, so that the Eulerian and Lagrangian infinitesimal strain tensors may be taken as equal. Thus

$$l_{ij} = \epsilon_{ij} \quad \text{or} \quad \mathbf{L} = \mathbf{E} \qquad (3.44)$$

if both the displacements and displacement gradients are sufficiently small.

3.8 RELATIVE DISPLACEMENTS. LINEAR ROTATION TENSOR. ROTATION VECTOR

In Fig. 3-3 the displacements of two neighboring particles are represented by the vectors $u_i^{(P_0)}$ and $u_i^{(Q_0)}$ (see also Fig. 3-2). The vector

$$du_i = u_i^{(Q_0)} - u_i^{(P_0)} \quad \text{or} \quad d\mathbf{u} = \mathbf{u}^{(Q_0)} - \mathbf{u}^{(P_0)} \qquad (3.45)$$

is called the *relative displacement vector* of the particle originally at Q_0 with respect to the particle originally at P_0. Assuming suitable continuity conditions on the displacement field, a Taylor series expansion for $u_i^{(P_0)}$ may be developed in the neighborhood of P_0. Neglecting higher-order terms in this expansion, the relative displacement vector can be written as

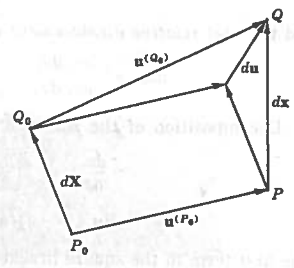

Fig. 3-3

$$du_i = (\partial u_i/\partial X_j)_{P_0}\, dX_j \quad \text{or} \quad d\mathbf{u} = (\mathbf{u}\nabla_{\mathbf{X}})_{P_0}\cdot d\mathbf{X} \qquad (3.46)$$

Here the parentheses on the partial derivatives are to emphasize the requirement that the derivatives are to be evaluated at point P_0. These derivatives are actually the components of the material displacement gradient. Equation (3.46) is the Lagrangian form of the relative displacement vector.

It is also useful to define the *unit relative displacement vector* du_i/dX in which dX is the magnitude of the differential distance vector dX_i. Accordingly if ν_i is a unit vector in the direction of dX_i so that $dX_i = \nu_i\, dX$, then

$$\frac{du_i}{dX} = \frac{\partial u_i}{\partial X_j}\frac{dX_j}{dX} = \frac{\partial u_i}{\partial X_j}\nu_j \quad \text{or} \quad \frac{d\mathbf{u}}{dX} = \mathbf{u}\nabla_{\mathbf{X}}\cdot\hat{\boldsymbol{\nu}} = \mathbf{J}\cdot\hat{\boldsymbol{\nu}} \qquad (3.47)$$

Since the material displacement gradient $\partial u_i/\partial X_j$ may be decomposed uniquely into a symmetric and an antisymmetric part, the relative displacement vector du_i may be written as

$$du_i = \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right)\right] dX_j$$

or

$$d\mathbf{u} = [\tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{X}} + \nabla_{\mathbf{X}}\mathbf{u}) + \tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{X}} - \nabla_{\mathbf{X}}\mathbf{u})]\cdot d\mathbf{X} \qquad (3.48)$$

The first term in the square brackets in (3.48) is recognized as the linear Lagrangian strain tensor l_{ij}. The second term is known as the *linear Lagrangian rotation tensor* and is denoted by

$$W_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right) \quad \text{or} \quad \mathbf{W} = \tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{X}} - \nabla_{\mathbf{X}}\mathbf{u}) \qquad (3.49)$$

In a displacement for which the strain tensor l_{ij} is identically zero in the vicinity of point P_0, the relative displacement at that point will be an infinitesimal rigid body rotation. This infinitesimal rotation may be represented by the *rotation vector*

$$w_i = \tfrac{1}{2}\epsilon_{ijk}W_{kj} \quad \text{or} \quad \mathbf{w} = \tfrac{1}{2}\nabla_{\mathbf{X}} \times \mathbf{u} \qquad (3.50)$$

in terms of which the relative displacement is given by the expression

$$du_i = \epsilon_{ijk}w_j\,dX_k \quad \text{or} \quad d\mathbf{u} = \mathbf{w} \times d\mathbf{X} \qquad (3.51)$$

The development of the Lagrangian description of the relative displacement vector, the linear rotation tensor and the linear rotation vector is paralleled completely by an analogous development for the Eulerian counterparts of these quantities. Accordingly the *Eulerian description* of the *relative displacement vector* is given by

$$du_i = \frac{\partial u_i}{\partial x_j}dx_j \quad \text{or} \quad d\mathbf{u} = \mathbf{K}\cdot d\mathbf{x} \qquad (3.52)$$

and the *unit relative displacement vector* by

$$du_i = \frac{\partial u_i}{\partial x_j}\frac{dx_j}{dx} = \frac{\partial u_i}{\partial x_j}\mu_j \quad \text{or} \quad \frac{d\mathbf{u}}{dx} = \mathbf{u}\nabla_{\mathbf{x}}\cdot\hat{\boldsymbol{\mu}} = \mathbf{K}\cdot\hat{\boldsymbol{\mu}} \qquad (3.53)$$

Decomposition of the Eulerian displacement gradient $\partial u_i/\partial x_j$ results in the expression

$$\frac{du_i}{dx} = \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\right] dx_j$$

or

$$d\mathbf{u} = [\tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{x}} + \nabla_{\mathbf{x}}\mathbf{u}) + \tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{x}} - \nabla_{\mathbf{x}}\mathbf{u})]\cdot d\mathbf{x} \qquad (3.54)$$

The first term in the square brackets of (3.54) is the Eulerian linear strain tensor ϵ_{ij}. The second term is the *linear Eulerian rotation tensor* and is denoted by

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \quad \text{or} \quad \boldsymbol{\Omega} = \tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{x}} - \nabla_{\mathbf{x}}\mathbf{u}) \qquad (3.55)$$

From (3.55), the *linear Eulerian rotation vector* is defined by

$$\omega_i = \tfrac{1}{2}\epsilon_{ijk}\omega_{kj} \quad \text{or} \quad \boldsymbol{\omega} = \tfrac{1}{2}\nabla_{\mathbf{x}} \times \mathbf{u} \qquad (3.56)$$

in terms of which the relative displacement is given by the expression

$$du_i = \epsilon_{ijk}\omega_j\,dx_k \quad \text{or} \quad d\mathbf{u} = \boldsymbol{\omega} \times d\mathbf{x} \qquad (3.57)$$

3.9 INTERPRETATION OF THE LINEAR STRAIN TENSORS

For small deformation theory, the finite Lagrangian strain tensor L_{ij} in (3.36) may be replaced by the linear Lagrangian strain tensor l_{ij}, and that expression may now be written

$$(dx)^2 - (dX)^2 = (dx - dX)(dx + dX) = 2l_{ij}\,dX_i\,dX_j$$

or

$$(dx)^2 - (dX)^2 = (dx - dX)(dx + dX) = d\mathbf{X}\cdot 2\mathbf{l}\cdot d\mathbf{X} \qquad (3.58)$$

Since $dx \sim dX$ for small deformations, this equation may be put into the form

$$\frac{dx - dX}{dX} = l_{ij}\frac{dX_i}{dX}\frac{dX_j}{dX} = l_{ij}\nu_i\nu_j \quad \text{or} \quad \frac{dx - dX}{dX} = \hat{\boldsymbol{\nu}}\cdot\mathbf{l}\cdot\hat{\boldsymbol{\nu}} \qquad (3.59)$$

The left-hand side of (3.59) is recognized as the change in length per unit original length of the differential element and is called the *normal strain* for the line element originally having direction cosines dX_i/dX.

When (3.59) is applied to the differential line element P_0Q_0, located with respect to the set of local axes at P_0 as shown in Fig. 3-4, the result will be the normal strain for that element. Because P_0Q_0 here lies along the X_2 axis,

$$dX_1/dX = dX_3/dX = 0, \quad dX_2/dX = 1$$

and therefore (3.59) becomes

$$\frac{dx - dX}{dX} = l_{22} = \frac{\partial u_2}{\partial X_2} \qquad (3.60)$$

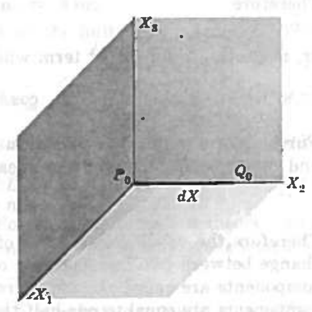

Fig. 3-4

Thus the normal strain for an element originally along the X_2 axis is seen to be the component l_{22}. Likewise for elements originally situated along the X_1 and X_3 axes, (3.59) yields normal strain values l_{11} and l_{33} respectively. In general, therefore, the diagonal terms of the linear strain tensor represent normal strains in the coordinate directions.

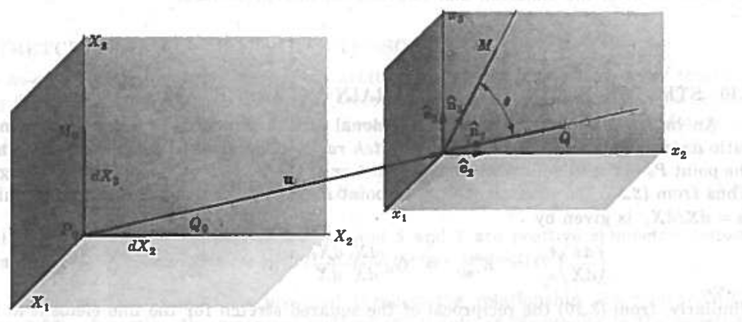

Fig. 3-5

The physical interpretation of the off-diagonal terms of l_{ij} may be obtained by a consideration of the line elements originally located along two of the coordinate axes. In Fig. 3-5 the line elements P_0Q_0 and P_0M_0 originally along the X_2 and X_3 axes, respectively, become after deformation the line elements PQ and PM with respect to the parallel set of local axes with origin at P. The original right angle between the line elements becomes the angle θ. From (3.46) and the assumption of small deformation theory, a first order approximation gives the unit vector at P in the direction of Q as

$$\hat{\mathbf{n}}_2 = \frac{\partial u_1}{\partial X_2}\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \frac{\partial u_3}{\partial X_2}\hat{\mathbf{e}}_3 \tag{3.61}$$

and, for the unit vector at P in the direction of M, as

$$\hat{\mathbf{n}}_3 = \frac{\partial u_1}{\partial X_3}\hat{\mathbf{e}}_1 + \frac{\partial u_2}{\partial X_3}\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \tag{3.62}$$

Therefore

$$\cos\theta = \hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_3 = \frac{\partial u_1}{\partial X_3}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} \tag{3.63}$$

or, neglecting the product term which is of higher order,

$$\cos\theta = \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} = 2l_{23} \tag{3.64}$$

Furthermore, taking the change in the right angle between the elements as $\gamma_{23} = \pi/2 - \theta$, and remembering that for the linear theory γ_{23} is very small, it follows that

$$\gamma_{23} \sim \sin\gamma_{23} = \sin(\pi/2 - \theta) = \cos\theta = 2l_{23} \tag{3.65}$$

Therefore the off-diagonal terms of the linear strain tensor represent one-half the angle change between two line elements originally at right angles to one another. These strain components are called *shearing strains*, and because of the factor 2 in (*3.65*) these tensor components are equal to one-half the familiar "engineering" shearing strains.

A development, essentially paralleling the one just presented for the interpretation of the components of l_{ij}, may also be made for the linear Eulerian strain tensor ϵ_{ij}. The essential difference in the derivations rests in the choice of line elements, which in the Eulerian description must be those that lie along the coordinate axes after deformation. The diagonal terms of ϵ_{ij} are the normal strains, and the off-diagonal terms the shearing strains. For those deformations in which the assumption $l_{ij} = \epsilon_{ij}$ is valid, no distinction is made between the Eulerian and Lagrangian interpretations.

3.10 STRETCH RATIO. FINITE STRAIN INTERPRETATION

An important measure of the extensional strain of a differential line element is the ratio dx/dX, known as the *stretch* or *stretch ratio*. This quantity may be defined at either the point P_0 in the undeformed configuration or at the point P in the deformed configuration. Thus from (*3.34*) the squared stretch at point P_0 for the line element along the unit vector $\hat{\mathbf{m}} = d\mathbf{X}/dX$, is given by

$$\left(\frac{dx}{dX}\right)^2_{P_0} = \Lambda^2_{(\hat{\mathbf{m}})} = G_{ij}\frac{dX_i}{dX}\frac{dX_j}{dX} \quad \text{or} \quad \Lambda^2_{(\hat{\mathbf{m}})} = \hat{\mathbf{m}}\cdot\mathbf{G}\cdot\hat{\mathbf{m}} \tag{3.66}$$

Similarly, from (*3.30*) the reciprocal of the squared stretch for the line element at P along the unit vector $\hat{\mathbf{n}} = d\mathbf{x}/dx$ is given by

$$\left(\frac{dX}{dx}\right)^2_{P} = \frac{1}{\lambda^2_{(\hat{\mathbf{n}})}} = C_{ij}\frac{dx_i}{dx}\frac{dx_j}{dx} \quad \text{or} \quad \frac{1}{\lambda^2_{(\hat{\mathbf{n}})}} = \hat{\mathbf{n}}\cdot\mathbf{C}\cdot\hat{\mathbf{n}} \tag{3.67}$$

For an element originally along the local X_2 axis shown in Fig. 3-4, $\hat{\mathbf{m}} \equiv \hat{\mathbf{e}}_2$ and therefore $dX_1/dX = dX_3/dX = 0$, $dX_2/dX = 1$ so that (*3.66*) yields for such an element

$$\Lambda^2_{(\hat{\mathbf{e}}_2)} = G_{22} = 1 + 2L_{22} \tag{3.68}$$

Similar results may be determined for $\Lambda^2_{(\hat{\mathbf{e}}_1)}$ and $\Lambda^2_{(\hat{\mathbf{e}}_3)}$.

For an element parallel to the x_2 axis after deformation, (*3.67*) yields the result

$$\frac{1}{\lambda^2_{(\hat{\mathbf{e}}_2)}} = 1 - 2E_{22} \tag{3.69}$$

with similar expressions for the quantities $1/\lambda^2_{(\hat{\mathbf{e}}_1)}$ and $1/\lambda^2_{(\hat{\mathbf{e}}_3)}$. In general, $\Lambda_{(\hat{\mathbf{e}}_2)}$ is not equal to $\lambda_{(\hat{\mathbf{e}}_2)}$ since the element originally along the X_2 axis will not likely lie along the x_2 axis after deformation.

The stretch ratio provides a basis for interpretation of the finite strain tensors. Thus the change of length per unit of original length is

$$\frac{dx - dX}{dX} = \frac{dx}{dX} - 1 = \Lambda_{(\hat{\mathbf{m}})} - 1 \tag{3.70}$$

and for the element P_0Q_0 along the X_2 axis (of Fig. 3-4), the *unit extension* is therefore

$$L_{(2)} = \Lambda_{(\hat{\mathbf{e}}_2)} - 1 = \sqrt{1 + 2L_{22}} - 1 \tag{3.71}$$

This result may also be derived directly from (*3.36*). For small deformation theory, (*3.71*) reduces to (*3.60*). Also, the unit extensions $L_{(1)}$ and $L_{(3)}$ are given by analogous equations in terms of L_{11} and L_{33} respectively.

For the two differential line elements shown in Fig. 3-5, the change in angle $\gamma_{23} = \pi/2 - \theta$ is given in terms of $\Lambda_{(\hat{\mathbf{e}}_2)}$ and $\Lambda_{(\hat{\mathbf{e}}_3)}$ by

$$\sin\gamma_{23} = \frac{2L_{23}}{\Lambda_{(\hat{\mathbf{e}}_2)}\Lambda_{(\hat{\mathbf{e}}_3)}} = \frac{2L_{23}}{\sqrt{1+2L_{22}}\sqrt{1+2L_{33}}} \tag{3.72}$$

When deformations are small, (*3.72*) reduces to (*3.65*).

3.11 STRETCH TENSORS. ROTATION TENSOR

The so-called *polar decomposition* of an arbitrary, nonsingular, second-order tensor is given by the product of a positive symmetric second-order tensor with an orthogonal second-order tensor. When such a multiplicative decomposition is applied to the deformation gradient $\mathbf{F}$, the result may be written

$$F_{ij} \equiv \frac{\partial x_i}{\partial X_j} = R_{ik}S_{kj} = T_{ik}R_{kj} \quad \text{or} \quad \mathbf{F} = \mathbf{R}\cdot\mathbf{S} = \mathbf{T}\cdot\mathbf{R} \tag{3.73}$$

in which $\mathbf{R}$ is the *orthogonal rotation tensor*, and $\mathbf{S}$ and $\mathbf{T}$ are positive symmetric tensors known as the *right stretch tensor* and *left stretch tensor* respectively.

The interpretation of (*3.73*) is provided through the relationship $dx_i = (\partial x_i/\partial X_j)\,dX_j$ given by (*3.33*). Inserting the inner products of (*3.73*) into (*3.33*) results in the equations

$$dx_i = R_{ik}S_{kj}\,dX_j = T_{ik}R_{kj}\,dX_j \quad \text{or} \quad d\mathbf{x} = \mathbf{R}\cdot\mathbf{S}\cdot d\mathbf{X} = \mathbf{T}\cdot\mathbf{R}\cdot d\mathbf{X} \tag{3.74}$$

From these expressions the deformation of dX_i into dx_i as illustrated in Fig. 3-2 may be given either of two physical interpretations. In the first form of the right hand side of (*3.74*), the deformation consists of a sequential stretching (by $\mathbf{S}$) and rotation to be followed by a rigid body displacement to the point P. In the second form, a rigid body translation to P is followed by a rotation and finally the stretching (by $\mathbf{T}$). The translation, of course, does not alter the vector components relative to the axes X_i and x_i.

3.12 TRANSFORMATION PROPERTIES OF STRAIN TENSORS

The various strain tensors L_{ij}, E_{ij}, l_{ij} and ϵ_{ij} defined respectively by *(3.37)*, *(3.39)*, *(3.42)* and *(3.43)* are all second-order Cartesian tensors as indicated by the two free indices in each. Accordingly for a set of rotated axes X_i' having the transformation matrix $[b_{ij}]$ with respect to the set of local unprimed axes X_i at point P_0 as shown in Fig. 3-6(*a*), the components of L_{ij}' and l_{ij}' are given by

$$L_{ij}' = b_{ip}b_{jq}L_{pq} \quad \text{or} \quad \mathbf{L}_G' = \mathbf{B}\cdot\mathbf{L}_G\cdot\mathbf{B}_c \tag{3.75}$$

and

$$l_{ij}' = b_{ip}b_{jq}l_{pq} \quad \text{or} \quad \mathbf{l}' = \mathbf{B}\cdot\mathbf{l}\cdot\mathbf{B}_c \tag{3.76}$$

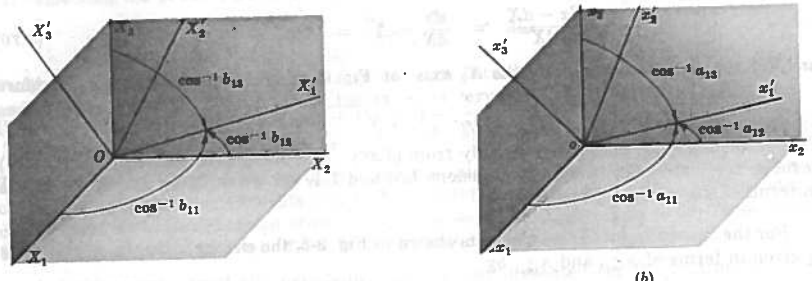

Fig. 3-6

Likewise, for the rotated axes x_i' having the transformation matrix $[a_{ij}]$ in Fig. 3-6(*b*), the components of E_{ij}' and e_{ij}' are given by

$$E_{ij}' = a_{ip}a_{jq}E_{pq} \quad \text{or} \quad \mathbf{E}_A' = \mathbf{A}\cdot\mathbf{E}_A\cdot\mathbf{A}_c \tag{3.77}$$

and

$$\epsilon_{ij}' = a_{ip}a_{jq}\epsilon_{pq} \quad \text{or} \quad \mathbf{E}' = \mathbf{A}\cdot\mathbf{E}\cdot\mathbf{A}_c \tag{3.78}$$

By analogy with the stress quadric described in Section 2.9, page 50, the *Lagrangian* and *Eulerian linear strain quadrics* may be given with reference to local Cartesian coordinates η_i and ζ_i at the points P_0 and P respectively as shown in Fig. 3-7. Thus the equation of the *Lagrangian strain quadric* is given by

$$l_{ij}\eta_i\eta_j = \pm h^2 \quad \text{or} \quad \boldsymbol{\eta}\cdot\mathbf{l}\cdot\boldsymbol{\eta} = \pm h^2 \tag{3.79}$$

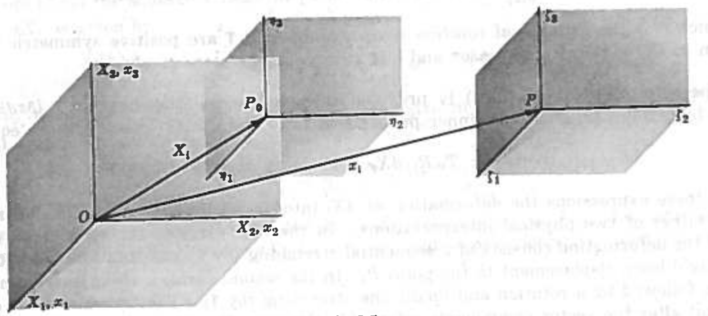

Fig. 3-7

and the equation of the *Eulerian strain quadric* is given by

$$\epsilon_{ij}\zeta_i\zeta_j = \pm g^2 \quad \text{or} \quad \boldsymbol{\zeta}\cdot\mathbf{E}\cdot\boldsymbol{\zeta} = \pm g^2 \tag{3.80}$$

Two important properties of the Lagrangian (Eulerian) linear strain quadric are:

1. The normal strain with respect to the original (final) length of a line element is inversely proportional to the distance squared from the origin of the quadric P_0 (P) to a point on its surface.
2. The relative displacement of the neighboring particle located at Q_0 (Q) per unit original (final) length is parallel to the normal of the quadric surface at the point of intersection with the line through P_0Q_0 (PQ).

Additional insight into the nature of local deformations in the neighborhood of P_0 is provided by defining the *strain ellipsoid* at that point. Thus for the undeformed continuum, the equation of the bounding surface of an infinitesimal sphere of radius R is given in terms of local material coordinates by *(3.28)* as

$$(dX)^2 = \delta_{ij}\,dX_i\,dX_j = R^2 \quad \text{or} \quad (dX)^2 = d\mathbf{X}\cdot\mathbf{I}\cdot d\mathbf{X} = R^2 \tag{3.81}$$

After deformation, the equation of the surface of the same material particles is given by *(3.30)* as

$$(dX)^2 = C_{ij}\,dx_i\,dx_j = R^2 \quad \text{or} \quad (dX)^2 = d\mathbf{x}\cdot\mathbf{C}\cdot d\mathbf{x} = R^2 \tag{3.82}$$

which describes an ellipsoid, known as the *material strain ellipsoid*. Therefore a spherical volume of the continuum in the undeformed state is changed into an ellipsoid at P_0 by the deformation. By comparison, an infinitesimal spherical volume at P in the deformed continuum began as an ellipsoidal volume element in the undeformed state. For a sphere of radius r at P, the equations for these surfaces in terms of local coordinates are given by *(3.32)* for the sphere as

$$(dx)^2 = \delta_{ij}\,dx_i\,dx_j = r^2 \quad \text{or} \quad (dx)^2 = d\mathbf{x}\cdot\mathbf{I}\cdot d\mathbf{x} = r^2 \tag{3.83}$$

and by *(3.34)* for the ellipsoid as

$$(dx)^2 = G_{ij}\,dX_i\,dX_j = r^2 \quad \text{or} \quad (dx)^2 = d\mathbf{X}\cdot\mathbf{G}\cdot d\mathbf{X} = r^2 \tag{3.84}$$

The ellipsoid of *(3.84)* is called the *spatial strain ellipsoid*. Such strain ellipsoids as described here are frequently known as *Cauchy strain ellipsoids*.

3.13 PRINCIPAL STRAINS. STRAIN INVARIANTS. CUBICAL DILATATION

The Lagrangian and Eulerian linear strain tensors are symmetric second-order Cartesian tensors, and accordingly the determination of their principal directions and principal strain values follows the standard development presented in Section 1.19, page 20. Physically, a principal direction of the strain tensor is one for which the orientation of an element at a given point is not altered by a pure strain deformation. The principal strain value is simply the unit relative displacement (normal strain) that occurs in the principal direction.

For the Lagrangian strain tensor l_{ij}, the unit relative displacement vector is given by *(3.47)*, which may be written

$$\frac{du_i}{dX} = (l_{ij} + W_{ij})\nu_j \quad \text{or} \quad \frac{d\mathbf{u}}{dX} = (\mathbf{L}+\mathbf{W})\cdot\hat{\boldsymbol{\nu}} \tag{3.85}$$

Calling $l_i^{(\hat{n})}$ the normal strain in the direction of the unit vector n_i, *(3.85)* yields for pure strain ($W_{ij} \equiv 0$) the relation

$$l_i^{(\hat{n})} = l_{ij}n_j \quad \text{or} \quad \mathbf{l}^{(\hat{n})} = \mathbf{L}\cdot\hat{\mathbf{n}} \tag{3.86}$$

If the direction n_i is a principal direction with a principal strain value l, then

$$l_i^{(\hat{n})} = l n_i = l \delta_{ij} n_j \quad \text{or} \quad \mathbf{l}^{(\hat{n})} = l \hat{\mathbf{n}} = l \mathbf{I} \cdot \hat{\mathbf{n}} \tag{3.87}$$

Equating the right-hand sides of (*3.86*) and (*3.87*) leads to the relationship

$$(l_{ij} - \delta_{ij} l) n_j = 0 \quad \text{or} \quad (\mathbf{L} - l\mathbf{I}) \cdot \hat{\mathbf{n}} = 0 \tag{3.88}$$

which together with the condition $n_i n_i = 1$ on the unit vectors n_i provide the necessary equations for determining the principal strain value l and its direction cosines n_i. Nontrivial solutions of (*3.88*) exist if and only if the determinant of coefficients vanishes. Therefore

$$|l_{ij} - \delta_{ij} l| = 0 \quad \text{or} \quad |\mathbf{L} - l\mathbf{I}| = 0 \tag{3.89}$$

which upon expansion yields the characteristic equation of l_{ij}, the cubic

$$l^3 - \text{I}_L l^2 + \text{II}_L l - \text{III}_L = 0 \tag{3.90}$$

where $\quad \text{I}_L = l_{ii} = \text{tr}\,\mathbf{L}, \quad \text{II}_L = \tfrac{1}{2}(l_{ii}l_{jj} - l_{ij}l_{ij}), \quad \text{III}_L = |l_{ij}| = \det \mathbf{L} \qquad (3.91)$

are the first, second and third *Lagrangian strain invariants* respectively. The roots of (*3.90*) are the principal strain values denoted by $l_{(1)}$, $l_{(2)}$ and $l_{(3)}$.

The first invariant of the Lagrangian strain tensor may be expressed in terms of the principal strains as

$$\text{I}_L = l_{ii} = l_{(1)} + l_{(2)} + l_{(3)} \tag{3.92}$$

and has an important physical interpretation. To see this, consider a differential rectangular parallelepiped whose edges are parallel to the principal strain directions as shown in Fig. 3-8. The change in volume per unit original volume of this element is called the *cubical dilatation* and is given by

$$D_0 = \frac{\Delta V_0}{V_0} = \frac{dX_1(1 + l_{(1)})\,dX_2(1 + l_{(2)})\,dX_3(1 + l_{(3)}) - dX_1\,dX_2\,dX_3}{dX_1\,dX_2\,dX_3} \tag{3.93}$$

For small strain theory, the first-order approximation of this ratio is the sum

$$D_0 = l_{(1)} + l_{(2)} + l_{(3)} = \text{I}_L \tag{3.94}$$

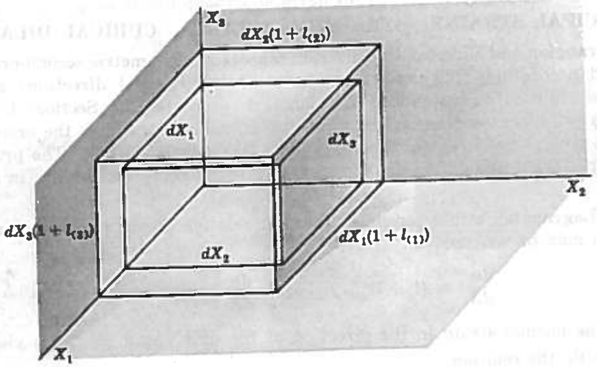

Fig. 3-8

With regard to the Eulerian strain tensor ϵ_{ij} and its associated unit relative displacement vector $\epsilon_i^{(\hat{n})}$, the principal directions and principal strain values $\epsilon_{(1)}, \epsilon_{(2)}, \epsilon_{(3)}$ are determined in exactly the same way as their Lagrangian counterparts. The Eulerian strain invariants may be expressed in terms of the principal strains as

$$\begin{aligned} \text{I}_E &= \epsilon_{(1)} + \epsilon_{(2)} + \epsilon_{(3)} \\ \text{II}_E &= \epsilon_{(1)}\epsilon_{(2)} + \epsilon_{(2)}\epsilon_{(3)} + \epsilon_{(3)}\epsilon_{(1)} \\ \text{III}_E &= \epsilon_{(1)}\epsilon_{(2)}\epsilon_{(3)} \end{aligned} \tag{3.95}$$

The cubical dilatation for the Eulerian description is given by

$$\Delta V/V = D = \epsilon_{(1)} + \epsilon_{(2)} + \epsilon_{(3)} \tag{3.96}$$

3.14 SPHERICAL AND DEVIATOR STRAIN TENSORS

The Lagrangian and Eulerian *linear* strain tensors may each be split into a *spherical* and *deviator* tensor in the same manner in which the stress tensor decomposition was carried out in Chapter 2. As before, if Lagrangian and Eulerian deviator tensor components are denoted by d_{ij} and e_{ij} respectively, the resolution expressions are

$$l_{ij} = d_{ij} + \delta_{ij}\frac{l_{kk}}{3} \quad \text{or} \quad \mathbf{L} = \mathbf{L}_D + \frac{\mathbf{I}(\text{tr}\,\mathbf{L})}{3} \tag{3.97}$$

and

$$\epsilon_{ij} = e_{ij} + \delta_{ij}\frac{\epsilon_{kk}}{3} \quad \text{or} \quad \mathbf{E} = \mathbf{E}_D + \frac{\mathbf{I}(\text{tr}\,\mathbf{E})}{3} \tag{3.98}$$

The deviator tensors are associated with shear deformation for which the cubical dilatation vanishes. Therefore it is not surprising that the first invariants d_{ii} and e_{ii} of the deviator strain tensors are identically zero.

3.15 PLANE STRAIN. MOHR'S CIRCLES FOR STRAIN

When one and only one of the principal strains at a point in a continuum is zero, a state of *plane strain* is said to exist at that point. In the Eulerian description (the Lagrangian description follows exactly the same pattern), if x_3 is taken as the direction of the zero principal strain, a state of plane strain parallel to the x_1x_2 plane exists and the linear strain tensor is given by

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad [\epsilon_{ij}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.99}$$

When x_1 and x_2 are also principal directions, the strain tensor has the form

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{(1)} & 0 & 0 \\ 0 & \epsilon_{(2)} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad [\epsilon_{ij}] = \begin{bmatrix} \epsilon_{(1)} & 0 & 0 \\ 0 & \epsilon_{(2)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.100}$$

In many books on "Strength of Materials" and "Elasticity", plane strain is referred to as *plane deformation* since the deformation field is identical in all planes perpendicular to the direction of the zero principal strain. For plane strain perpendicular to the x_3 axis, the displacement vector may be taken as a function of x_1 and x_2 only. The appropriate displacement components for this case of plane strain are designated by

$$u_1 = u_1(x_1, x_2)$$
$$u_2 = u_2(x_1, x_2) \qquad (3.101)$$
$$u_3 = C \text{ (a constant, usually taken as zero)}$$

Inserting these expressions into the definition of ϵ_{ij} given by (3.43) produces the plane strain tensor in the same form shown in (3.99).

A graphical description of the state of strain at a point is provided by the *Mohr's circles for strain* in a manner exactly like that presented in Chapter 2 for the Mohr's circles for stress. For this purpose the strain tensor is often displayed in the form

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{11} & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\gamma_{12} & \epsilon_{22} & \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{13} & \frac{1}{2}\gamma_{23} & \epsilon_{33} \end{pmatrix} \qquad (3.102)$$

Here the γ_{ij} (with $i \neq j$) are the so-called "engineering" shear strain components, which are twice the tensorial shear strain components.

The state of strain at an unloaded point on the bounding surface of a continuum body is locally plane strain. Frequently in experimental studies involving strain measurements at such a surface point, Mohr's strain circles are useful for reporting the observed data. Usually three normal strains are measured at the given point by means of a strain rosette, and the Mohr's circles diagram constructed from these. Corresponding to the plane stress Mohr's circles, a typical case of plane strain diagram is shown in Fig. 3-9. The principal normal strains are labeled as such in the diagram, and the maximum shear strain values are represented by points D and E.

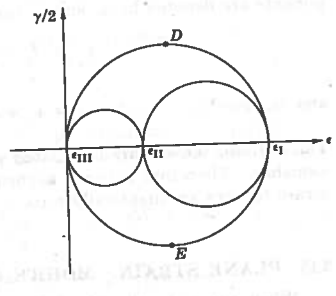

Fig. 3-9

3.16 COMPATIBILITY EQUATIONS FOR LINEAR STRAINS

If the strain components ϵ_{ij} are given explicitly as functions of the coordinates, the six independent equations (3.43)

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

may be viewed as a system of six partial differential equations for determining the three displacement components u_i. The system is over-determined and will not, in general, possess a solution for an arbitrary choice of the strain components ϵ_{ij}. Therefore if the displacement components u_i are to be single-valued and continuous, some conditions must be imposed upon the strain components. The necessary and sufficient conditions for such a displacement field are expressed by the equations

$$\frac{\partial^2 \epsilon_{ij}}{\partial x_k \partial x_m} + \frac{\partial^2 \epsilon_{km}}{\partial x_i \partial x_j} - \frac{\partial^2 \epsilon_{ik}}{\partial x_j \partial x_m} - \frac{\partial^2 \epsilon_{jm}}{\partial x_i \partial x_k} = 0 \qquad (3.103)$$

There are eighty-one equations in all in (3.103) but only six are distinct. These six written in explicit and symbolic form appear as

$$\left.\begin{aligned}
&1.\quad \frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} \\
&2.\quad \frac{\partial^2 \epsilon_{22}}{\partial x_3^2} + \frac{\partial^2 \epsilon_{33}}{\partial x_2^2} = 2\frac{\partial^2 \epsilon_{23}}{\partial x_2 \partial x_3} \\
&3.\quad \frac{\partial^2 \epsilon_{33}}{\partial x_1^2} + \frac{\partial^2 \epsilon_{11}}{\partial x_3^2} = 2\frac{\partial^2 \epsilon_{31}}{\partial x_3 \partial x_1} \\
&4.\quad \frac{\partial}{\partial x_1}\left(-\frac{\partial \epsilon_{23}}{\partial x_1} + \frac{\partial \epsilon_{31}}{\partial x_2} + \frac{\partial \epsilon_{12}}{\partial x_3}\right) = \frac{\partial^2 \epsilon_{11}}{\partial x_2 \partial x_3} \\
&5.\quad \frac{\partial}{\partial x_2}\left(\frac{\partial \epsilon_{23}}{\partial x_1} - \frac{\partial \epsilon_{31}}{\partial x_2} + \frac{\partial \epsilon_{12}}{\partial x_3}\right) = \frac{\partial^2 \epsilon_{22}}{\partial x_3 \partial x_1} \\
&6.\quad \frac{\partial}{\partial x_3}\left(\frac{\partial \epsilon_{23}}{\partial x_1} + \frac{\partial \epsilon_{31}}{\partial x_2} - \frac{\partial \epsilon_{12}}{\partial x_3}\right) = \frac{\partial^2 \epsilon_{33}}{\partial x_1 \partial x_2}
\end{aligned}\right\} \text{ or } \nabla_x \times \mathbf{E} \times \nabla_x = 0 \qquad (3.104)$$

Compatibility equations in terms of the Lagrangian linear strain tensor l_{ij} may also be written down by an obvious correspondence to the Eulerian form given above. For plane strain parallel to the x_1x_2 plane, the six equations in (3.104) reduce to the single equation

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} \quad \text{or} \quad \nabla_x \times \mathbf{E} \times \nabla_x = 0 \qquad (3.105)$$

where $\mathbf{E}$ is of the form given by (3.99).

Solved Problems

DISPLACEMENT AND DEFORMATION (Sec. 3.1-3.5)

3.1. With respect to superposed material axes X_i and spatial axes x_i, the displacement field of a continuum body is given by $x_1 = X_1$, $x_2 = X_2 + AX_3$, $x_3 = X_3 + AX_2$ where A is a constant. Determine the displacement vector components in both the material and spatial forms.

From (3.13) directly, the displacement components in material form are $u_1 = x_1 - X_1 = 0$, $u_2 = x_2 - X_2 = AX_3$, $u_3 = x_3 - X_3 = AX_2$. Inverting the given displacement relations to obtain $X_1 = x_1$, $X_2 = (x_2 - Ax_3)/(1-A^2)$, $X_3 = (x_3 - Ax_2)/(1-A^2)$, the spatial components of $\mathbf{u}$ are $u_1 = 0$, $u_2 = A(x_3 - Ax_2)/(1-A^2)$, $u_3 = A(x_2 - Ax_3)/(1-A^2)$.

From these results it is noted that the originally straight line of material particles expressed by $X_1 = 0$, $X_2 + X_3 = 1/(1+A)$ occupies the location $x_1 = 0$, $x_2 + x_3 = 1$ after displacement. Likewise the particle line $X_1 = 0$, $X_2 = X_3$ becomes after displacement $x_1 = 0$, $x_2 = x_3$. (Interpret the physical meaning of this.)

3.2. For the displacement field of Problem 3.1 determine the displaced location of the material particles which originally comprise (*a*) the plane circular surface $X_1 = 0$, $X_2^2 + X_3^2 = 1/(1-A^2)$, (*b*) the infinitesimal cube with edges along the coordinate axes of length $dX_i = dX$. Sketch the displaced configurations for (*a*) and (*b*) if $A = \frac{1}{2}$.

(*a*) By the direct substitutions $X_2 = (x_2 - Ax_3)/(1-A^2)$ and $X_3 = (x_3 - Ax_2)/(1-A^2)$, the circular surface becomes the elliptical surface $(1+A^2)x_2^2 - 4Ax_2x_3 + (1+A^2)x_3^2 = (1-A^2)$. For $A = \frac{1}{2}$, this is bounded by the ellipse $5x_2^2 - 8x_2x_3 + 5x_3^2 = 3$ which when referred to its principal axes x_i^* (at 45° with x_i, $i = 2, 3$) has the equation $x_2^{*2} + 9x_3^{*2} = 3$. Fig. 3-10 below shows this displacement pattern.