Chapter 2

Analysis of Stress

2.1 THE CONTINUUM CONCEPT

The molecular nature of the structure of matter is well established. In numerous investigations of material behavior, however, the individual molecule is of no concern and only the behavior of the material as a whole is deemed important. For these cases the observed macroscopic behavior is usually explained by disregarding molecular considerations and, instead, by assuming the material to be continuously distributed throughout its volume and to completely fill the space it occupies. This *continuum concept* of matter is the fundamental postulate of Continuum Mechanics. Within the limitations for which the continuum assumption is valid, this concept provides a framework for studying the behavior of solids, liquids and gases alike.

Adoption of the continuum viewpoint as the basis for the mathematical description of material behavior means that field quantities such as stress and displacement are expressed as piecewise continuous functions of the space coordinates and time.

2.2 HOMOGENEITY. ISOTROPY. MASS-DENSITY

A *homogeneous* material is one having identical properties at all points. With respect to some property, a material is *isotropic* if that property is the same in all directions at a point. A material is called *anisotropic* with respect to those properties which are directional at a point.

The concept of *density* is developed from the *mass-volume ratio* in the neighborhood of a point in the continuum. In Fig. 2-1 the mass in the small element of volume ΔV is denoted by ΔM. The *average density* of the material within ΔV is therefore

$$\rho_{(av)} = \frac{\Delta M}{\Delta V} \qquad (2.1)$$

The *density* at some interior point P of the volume element ΔV is given mathematically in accordance with the continuum concept by the limit,

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta M}{\Delta V} = \frac{dM}{dV} \qquad (2.2)$$

Mass-density ρ is a scalar quantity.

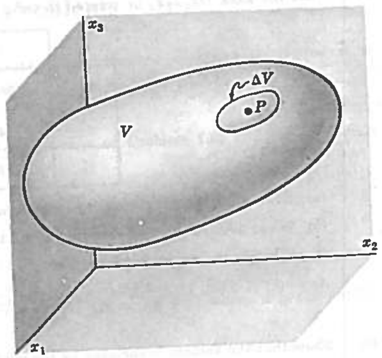

Fig. 2-1

2.3 BODY FORCES. SURFACE FORCES

Forces are vector quantities which are best described by intuitive concepts such as push or pull. Those forces which act on all elements of volume of a continuum are known as *body forces*. Examples are gravity and inertia forces. These forces are represented by the symbol b_i (force per unit mass), or as p_i (force per unit volume). They are related through the density by the equation

$$\rho b_i = p_i \quad \text{or} \quad \rho \mathbf{b} = \mathbf{p} \qquad (2.3)$$

Those forces which act on a surface element, whether it is a portion of the bounding surface of the continuum or perhaps an arbitrary internal surface, are known as *surface forces*. These are designated by f_i (force per unit area). Contact forces between bodies are a type of surface forces.

2.4 CAUCHY'S STRESS PRINCIPLE. THE STRESS VECTOR

A material continuum occupying the region R of space, and subjected to surface forces f_i and body forces b_i, is shown in Fig. 2-2. As a result of forces being transmitted from one portion of the continuum to another, the material within an arbitrary volume V enclosed by the surface S interacts with the material outside of this volume. Taking n_i as the outward unit normal at point P of a small element of surface ΔS of S, let Δf_i be the resultant force exerted across ΔS upon the material within V by the material outside of V. Clearly the force element Δf_i will depend upon the choice of ΔS and upon n_i. It should also be noted that the distribution of force on ΔS is not necessarily uniform. Indeed the force distribution is, in general, equipollent to a force and a moment at P, as shown in Fig. 2-2 by the vectors Δf_i and ΔM_i.

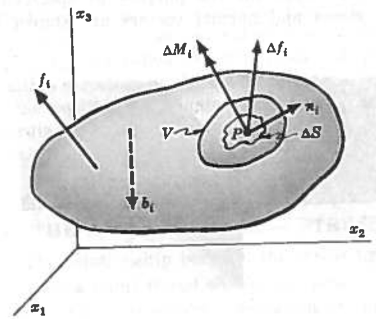

Fig. 2-2

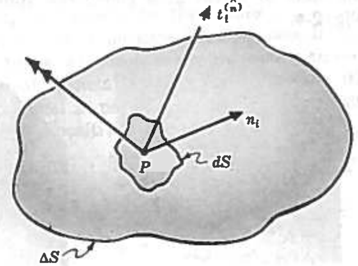

Fig. 2-3

The average force per unit area on ΔS is given by $\Delta f_i/\Delta S$. The *Cauchy stress principle* asserts that this ratio $\Delta f_i/\Delta s$ tends to a definite limit df_i/dS as ΔS approaches zero at the point P, while at the same time the moment of Δf_i about the point P vanishes in the limiting process. The resulting vector df_i/dS (force per unit area) is called the *stress vector* $t_i^{(\hat{n})}$ and is shown in Fig. 2-3. If the moment at P were not to vanish in the limiting process, a *couple-stress vector*, shown by the double-headed arrow in Fig. 2-3, would also be defined at the point. One branch of the theory of elasticity considers such couple stresses but they are not considered in this text.

Mathematically the stress vector is defined by

$$t_i^{(\hat{n})} = \lim_{\Delta S \to 0} \frac{\Delta f_i}{\Delta S} = \frac{df_i}{dS} \quad \text{or} \quad \mathbf{t}^{(\hat{n})} = \lim_{\Delta S \to 0} \frac{\Delta \mathbf{f}}{\Delta S} = \frac{d\mathbf{f}}{dS} \tag{2.4}$$

The notation $t_i^{(\hat{n})}$ (or $\mathbf{t}^{(\hat{n})}$) is used to emphasize the fact that the stress vector at a given point P in the continuum depends explicitly upon the particular surface element ΔS chosen there, as represented by the unit normal n_i (or $\hat{\mathbf{n}}$). For some differently oriented surface element, having a different unit normal, the associated stress vector at P will also be different. The stress vector arising from the action across ΔS at P of the material within V upon the material outside is the vector $-t_i^{(\hat{n})}$. Thus by Newton's law of action and reaction,

$$-t_i^{(\hat{n})} = t_i^{(-\hat{n})} \quad \text{or} \quad -\mathbf{t}^{(\hat{n})} = \mathbf{t}^{(-\hat{n})} \tag{2.5}$$

The *stress vector* is very often referred to as the *traction vector*.

2.5 STATE OF STRESS AT A POINT. STRESS TENSOR

At an arbitrary point P in a continuum, Cauchy's stress principle associates a stress vector $t_i^{(\hat{n})}$ with each unit normal vector n_i, representing the orientation of an infinitesimal surface element having P as an interior point. This is illustrated in Fig. 2-3. The totality of all possible pairs of such vectors $t_i^{(\hat{n})}$ and n_i at P defines the *state of stress* at that point. Fortunately it is not necessary to specify every pair of stress and normal vectors to completely describe the state of stress at a given point. This may be accomplished by giving the stress vector on each of three mutually perpendicular planes at P. Coordinate transformation equations then serve to relate the stress vector on any other plane at the point to the given three.

Adopting planes perpendicular to the coordinate axes for the purpose of specifying the state of stress at a point, the appropriate stress and normal vectors are shown in Fig. 2-4.

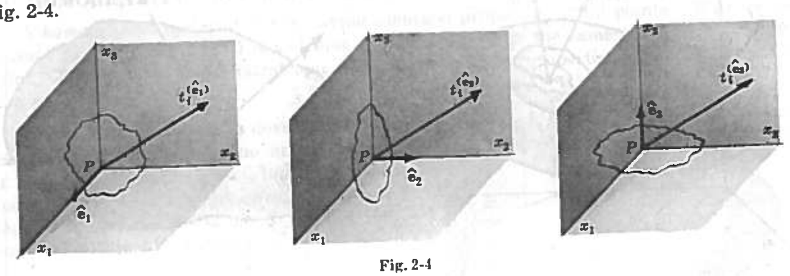

Fig. 2-4

For convenience, the three separate diagrams in Fig. 2-4 are often combined into a single schematic representation as shown in Fig. 2-5 below.

Each of the three coordinate-plane stress vectors may be written according to (*1.69*) in terms of its Cartesian components as

$$\begin{aligned} \mathbf{t}^{(\hat{e}_1)} &= t_1^{(\hat{e}_1)}\hat{\mathbf{e}}_1 + t_2^{(\hat{e}_1)}\hat{\mathbf{e}}_2 + t_3^{(\hat{e}_1)}\hat{\mathbf{e}}_3 = t_j^{(\hat{e}_1)}\hat{\mathbf{e}}_j \\ \mathbf{t}^{(\hat{e}_2)} &= t_1^{(\hat{e}_2)}\hat{\mathbf{e}}_1 + t_2^{(\hat{e}_2)}\hat{\mathbf{e}}_2 + t_3^{(\hat{e}_2)}\hat{\mathbf{e}}_3 = t_j^{(\hat{e}_2)}\hat{\mathbf{e}}_j \\ \mathbf{t}^{(\hat{e}_3)} &= t_1^{(\hat{e}_3)}\hat{\mathbf{e}}_1 + t_2^{(\hat{e}_3)}\hat{\mathbf{e}}_2 + t_3^{(\hat{e}_3)}\hat{\mathbf{e}}_3 = t_j^{(\hat{e}_3)}\hat{\mathbf{e}}_j \end{aligned} \tag{2.6}$$

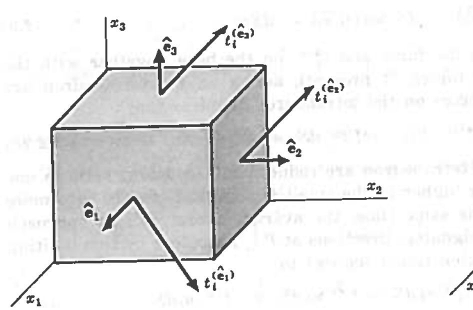

Fig. 2-5

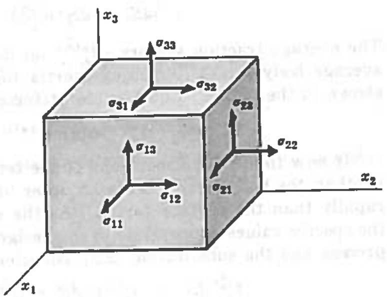

Fig. 2-6

The nine stress vector components,

$$t_j^{(\hat{e}_i)} \equiv \sigma_{ij} \tag{2.7}$$

are the components of a second-order Cartesian tensor known as the *stress tensor*. The equivalent stress dyadic is designated by Σ, so that explicit component and matrix representations of the stress tensor, respectively, take the forms

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \quad \text{or} \quad [\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{2.8}$$

Pictorially, the stress tensor components may be displayed with reference to the coordinate planes as shown in Fig. 2-6. The components perpendicular to the planes ($\sigma_{11}, \sigma_{22}, \sigma_{33}$) are called *normal stresses*. Those acting in (tangent to) the planes ($\sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{23}, \sigma_{31}, \sigma_{32}$) are called *shear stresses*. A stress component is *positive* when it acts in the positive direction of the coordinate axes, and on a plane whose outer normal points in one of the positive coordinate directions. The component σ_{ij} acts in the direction of the jth coordinate axis and on the plane whose outward normal is parallel to the ith coordinate axis. The stress components shown in Fig. 2-6 are all positive.

2.6 THE STRESS TENSOR — STRESS VECTOR RELATIONSHIP

The relationship between the stress tensor σ_{ij} at a point P and the stress vector $t_i^{(\hat{n})}$ on a plane of arbitrary orientation at that point may be established through the force equilibrium or momentum balance of a small tetrahedron of the continuum, having its vertex at P. The base of the tetrahedron is taken perpendicular to n_i, and the three faces are taken perpendicular to the coordinate planes as shown by Fig. 2-7. Designating the area of the base ABC as dS, the areas of the faces are the projected areas, $dS_1 = dS\,n_1$ for face CPB, $dS_2 = dS\,n_2$ for face APC, $dS_3 = dS\,n_3$ for face BPA or

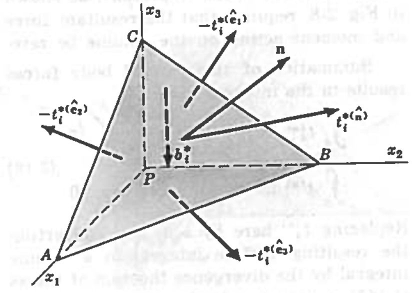

Fig. 2-7

$$dS_i = dS\,(\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_i) = dS\cos(\hat{\mathbf{n}},\hat{\mathbf{e}}_i) = dS\,n_i \qquad (2.9)$$

The average traction vectors $-t_i^{*(\hat{e}_i)}$ on the faces and $t_i^{*(\hat{n})}$ on the base, together with the average body forces (including inertia forces, if present), acting on the tetrahedron are shown in the figure. Equilibrium of forces on the tetrahedron requires that

$$t_i^{*(\hat{n})}\,dS - t_i^{*(\hat{e}_1)}\,dS_1 - t_i^{*(\hat{e}_2)}\,dS_2 - t_i^{*(\hat{e}_3)}\,dS_3 + \rho b_i^*\,dV = 0 \qquad (2.10)$$

If now the linear dimensions of the tetrahedron are reduced in a constant ratio to one another, the body forces, being an order higher in the small dimensions, tend to zero more rapidly than the surface forces. At the same time, the average stress vectors approach the specific values appropriate to the designated directions at P. Therefore by this limiting process and the substitution (*2.9*), equation (*2.10*) reduces to

$$t_i^{(\hat{n})}\,dS = t_i^{(\hat{e}_1)}n_1\,dS + t_i^{(\hat{e}_2)}n_2\,dS + t_i^{(\hat{e}_3)}n_3\,dS = t_i^{(\hat{e}_j)}n_j\,dS \qquad (2.11)$$

Cancelling the common factor dS and using the identity $t_i^{(\hat{e}_j)} \equiv \sigma_{ji}$, (*2.11*) becomes

$$t_i^{(\hat{n})} = \sigma_{ji}n_j \quad \text{or} \quad \mathbf{t}^{(\hat{n})} = \hat{\mathbf{n}}\cdot\mathbf{\Sigma} \qquad (2.12)$$

Equation (*2.12*) is also often expressed in the matrix form

$$[t_{1i}^{(\hat{n})}] = [n_{1k}][\sigma_{ki}] \qquad (2.13)$$

which is written explicitly

$$[t_1^{(\hat{n})}, t_2^{(\hat{n})}, t_3^{(\hat{n})}] = [n_1, n_2, n_3]\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix} \qquad (2.14)$$

The matrix form (*2.14*) is equivalent to the component equations

$$\begin{aligned} t_1^{(\hat{n})} &= n_1\sigma_{11} + n_2\sigma_{21} + n_3\sigma_{31}\\ t_2^{(\hat{n})} &= n_1\sigma_{12} + n_2\sigma_{22} + n_3\sigma_{32}\\ t_3^{(\hat{n})} &= n_1\sigma_{13} + n_2\sigma_{23} + n_3\sigma_{33}\end{aligned} \qquad (2.15)$$

2.7 FORCE AND MOMENT EQUILIBRIUM. STRESS TENSOR SYMMETRY

Equilibrium of an arbitrary volume V of a continuum, subjected to a system of surface forces $t_i^{(\hat{n})}$ and body forces b_i (including inertia forces, if present) as shown in Fig. 2-8, requires that the resultant force and moment acting on the volume be zero.

Summation of surface and body forces results in the integral relation,

$$\int_S t_i^{(\hat{n})}\,dS + \int_V \rho b_i\,dV = 0$$

or $\qquad (2.16)$

$$\int_S \mathbf{t}^{(\hat{n})}\,dS + \int_V \rho\mathbf{b}\,dV = 0$$

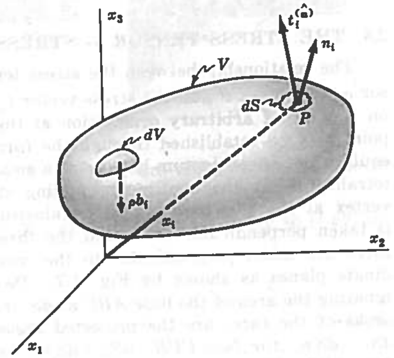

Fig. 2-8

Replacing $t_i^{(\hat{n})}$ here by $\sigma_{ji}n_j$ and converting the resulting surface integral to a volume integral by the divergence theorem of Gauss (*1.157*), equation (*2.16*) becomes

$$\int_V (\sigma_{ji,j} + \rho b_i)\,dV = 0 \quad \text{or} \quad \int_V (\nabla\cdot\mathbf{\Sigma} + \rho\mathbf{b})\,dV = 0 \qquad (2.17)$$

Since the volume V is arbitrary, the integrand in (*2.17*) must vanish, so that

$$\sigma_{ji,j} + \rho b_i = 0 \quad \text{or} \quad \nabla\cdot\mathbf{\Sigma} + \rho\mathbf{b} = 0 \qquad (2.18)$$

which are called the *equilibrium equations*.

In the absence of distributed moments or couple-stresses, the equilibrium of moments about the origin requires that

$$\int_S \epsilon_{ijk}x_j t_k^{(\hat{n})}\,dS + \int_V \epsilon_{ijk}x_j\rho b_k\,dV = 0$$

or $\qquad (2.19)$

$$\int_S \mathbf{x}\times\mathbf{t}^{(\hat{n})}\,dS + \int_V \mathbf{x}\times\rho\mathbf{b}\,dV = 0$$

in which x_i is the position vector of the elements of surface and volume. Again, making the substitution $t_i^{(\hat{n})} = \sigma_{ji}n_j$, applying the theorem of Gauss and using the result expressed in (*2.18*), the integrals of (*2.19*) are combined and reduced to

$$\int_V \epsilon_{ijk}\sigma_{jk}\,dV = 0 \quad \text{or} \quad \int_V \mathbf{\Sigma}_v\,dV = 0 \qquad (2.20)$$

For the arbitrary volume V, (*2.20*) requires

$$\epsilon_{ijk}\sigma_{jk} = 0 \quad \text{or} \quad \mathbf{\Sigma}_v = 0 \qquad (2.21)$$

Equation (*2.21*) represents the equations $\sigma_{12} = \sigma_{21}$, $\sigma_{23} = \sigma_{32}$, $\sigma_{13} = \sigma_{31}$, or in all

$$\sigma_{ij} = \sigma_{ji} \qquad (2.22)$$

which shows that the *stress tensor is symmetric*. In view of (*2.22*), the equilibrium equations (*2.18*) are often written

$$\sigma_{ij,j} + \rho b_i = 0 \qquad (2.23)$$

which appear in expanded form as

$$\begin{aligned} \frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + \frac{\partial\sigma_{13}}{\partial x_3} + \rho b_1 &= 0\\ \frac{\partial\sigma_{21}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{23}}{\partial x_3} + \rho b_2 &= 0\\ \frac{\partial\sigma_{31}}{\partial x_1} + \frac{\partial\sigma_{32}}{\partial x_2} + \frac{\partial\sigma_{33}}{\partial x_3} + \rho b_3 &= 0\end{aligned} \qquad (2.24)$$

2.8 STRESS TRANSFORMATION LAWS

At the point P let the rectangular Cartesian coordinate systems $Px_1x_2x_3$ and $Px_1'x_2'x_3'$ of Fig. 2-9 be related to one another by the table of direction cosines

	x_1	x_2	x_3
x_1'	a_{11}	a_{12}	a_{13}
x_2'	a_{21}	a_{22}	a_{23}
x_3'	a_{31}	a_{32}	a_{33}

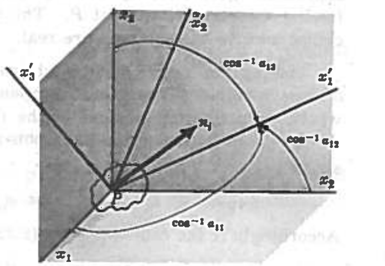

Fig. 2-9

or by the equivalent alternatives, the transformation matrix $[a_{ij}]$, or the transformation dyadic

$$\mathbf{A} = a_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j \qquad (2.25)$$

According to the transformation law for Cartesian tensors of order one (*1.93*), the components of the stress vector $t_i^{(\hat{n})}$ referred to the unprimed axes are related to the primed axes components $t_i'^{(\hat{n})}$ by the equation

$$t_i'^{(\hat{n})} = a_{ij}t_j^{(\hat{n})} \quad \text{or} \quad \mathbf{t}'^{(\hat{n})} = \mathbf{A}\cdot\mathbf{t}^{(\hat{n})} \qquad (2.26)$$

Likewise, by the transformation law (*1.102*) for second-order Cartesian tensors, the stress tensor components in the two systems are related by

$$\sigma'_{ij} = a_{ip}a_{jq}\sigma_{pq} \quad \text{or} \quad \mathbf{\Sigma}' = \mathbf{A}\cdot\mathbf{\Sigma}\cdot\mathbf{A}_c \qquad (2.27)$$

In matrix form, the stress vector transformation is written

$$[t_{i1}'^{(\hat{n})}] = [a_{ij}][t_{j1}^{(\hat{n})}] \qquad (2.28)$$

and the stress tensor transformation as

$$[\sigma'_{ij}] = [a_{ip}][\sigma_{pq}][a_{qj}] \qquad (2.29)$$

Explicitly, the matrix multiplications in (*2.28*) and (*2.29*) are given respectively by

$$\begin{bmatrix} t_1'^{(\hat{n})} \\ t_2'^{(\hat{n})} \\ t_3'^{(\hat{n})} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} t_1^{(\hat{n})} \\ t_2^{(\hat{n})} \\ t_3^{(\hat{n})} \end{bmatrix} \qquad (2.30)$$

and

$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \qquad (2.31)$$

2.9 STRESS QUADRIC OF CAUCHY

At the point P in a continuum, let the stress tensor have the values σ_{ij} when referred to directions parallel to the local Cartesian axes $P\zeta_1\zeta_2\zeta_3$ shown in Fig. 2-10. The equation

$$\sigma_{ij}\zeta_i\zeta_j = \pm k^2 \quad \text{(a constant)} \qquad (2.32)$$

represents geometrically similar quadric surfaces having a common center at P. The plus or minus choice assures the surfaces are real.

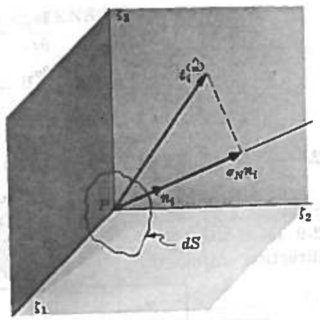

Fig. 2-10

The position vector $\mathbf{r}$ of an arbitrary point lying on the quadric surface has components $\zeta_i = rn_i$, where n_i is the unit normal in the direction of $\mathbf{r}$. At the point P the normal component $\sigma_N n_i$ of the stress vector $t_i^{(\hat{n})}$ has a magnitude

$$\sigma_N = t_i^{(\hat{n})}n_i = \mathbf{t}^{(\hat{n})}\cdot\mathbf{n} = \sigma_{ij}n_in_j \qquad (2.33)$$

Accordingly if the constant k^2 of (*2.32*) is set equal to $\sigma_N r^2$, the resulting quadric

$$\sigma_{ij}\zeta_i\zeta_j = \pm\sigma_N r^2 \qquad (2.34)$$

is called the ***stress quadric of Cauchy***. From this definition it follows that the magnitude σ_N of the normal stress component on the surface element dS perpendicular to the position vector $\mathbf{r}$ of a point on Cauchy's stress quadric, is inversely proportional to r^2, i.e. $\sigma_N = \pm k^2/r^2$. Furthermore it may be shown that the stress vector $t_i^{(\hat{n})}$ acting on dS at P is parallel to the normal of the tangent plane of the Cauchy quadric at the point identified by $\mathbf{r}$.

2.10 PRINCIPAL STRESSES. STRESS INVARIANTS. STRESS ELLIPSOID

At the point P for which the stress tensor components are σ_{ij}, the equation (*2.12*), $t_i^{(\hat{n})} = \sigma_{ji}n_j$, associates with each direction n_i a stress vector $t_i^{(\hat{n})}$. Those directions for which $t_i^{(\hat{n})}$ and n_i are collinear as shown in Fig. 2-11 are called *principal stress directions*. For a principal stress direction,

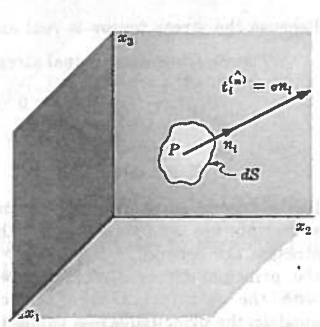

Fig. 2-11

$$t_i^{(\hat{n})} = \sigma n_i \quad \text{or} \quad \mathbf{t}^{(\hat{n})} = \sigma\hat{\mathbf{n}} \qquad (2.35)$$

in which σ, the magnitude of the stress vector, is called a *principal stress value*. Substituting (*2.35*) into (*2.12*) and making use of the identities $n_i = \delta_{ij}n_j$ and $\sigma_{ij} = \sigma_{ji}$, results in the equations

$$(\sigma_{ij} - \delta_{ij}\sigma)n_j = 0 \quad \text{or} \quad (\mathbf{\Sigma} - \mathbf{I}\sigma)\cdot\hat{\mathbf{n}} = 0 \qquad (2.36)$$

In the three equations (*2.36*), there are four unknowns, namely, the three direction cosines n_i and the principal stress value σ.

For solutions of (*2.36*) other than the trivial one $n_j = 0$, the determinant of coefficients, $|\sigma_{ij} - \delta_{ij}\sigma|$, must vanish. Explicitly,

$$|\sigma_{ij} - \delta_{ij}\sigma| = 0 \quad \text{or} \quad \begin{vmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{vmatrix} = 0 \qquad (2.37)$$

which upon expansion yields the cubic polynomial in σ,

$$\sigma^3 - \mathrm{I}_\Sigma\sigma^2 + \mathrm{II}_\Sigma\sigma - \mathrm{III}_\Sigma = 0 \qquad (2.38)$$

where

$$\mathrm{I}_\Sigma = \sigma_{ii} = \operatorname{tr}\mathbf{\Sigma} \qquad (2.39)$$

$$\mathrm{II}_\Sigma = \tfrac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}) \qquad (2.40)$$

$$\mathrm{III}_\Sigma = |\sigma_{ij}| = \det\mathbf{\Sigma} \qquad (2.41)$$

are known respectively as the *first*, *second* and *third stress invariants*.

The three roots of (*2.38*), $\sigma_{(1)}, \sigma_{(2)}, \sigma_{(3)}$ are the three principal stress values. Associated with each principal stress $\sigma_{(k)}$, there is a principal stress direction for which the direction cosines $n_i^{(k)}$ are solutions of the equations

$$(\sigma_{ij} - \sigma_{(k)}\delta_{ij})n_j^{(k)} = 0 \quad \text{or} \quad (\mathbf{\Sigma} - \sigma_{(k)}\mathbf{I})\cdot\hat{\mathbf{n}}^{(k)} = 0 \quad (k = 1,2,3) \qquad (2.42)$$

In (*2.42*) letter subscripts or superscripts enclosed by parentheses are merely labels and as such do not participate in any summation process. The expanded form of (*2.42*) for the *second* principal direction, for example, is therefore

$$(\sigma_{11} - \sigma_{(2)})n_1^{(2)} + \sigma_{12}n_2^{(2)} + \sigma_{13}n_3^{(2)} = 0$$
$$\sigma_{21}n_1^{(2)} + (\sigma_{22} - \sigma_{(2)})n_2^{(2)} + \sigma_{23}n_3^{(2)} = 0 \qquad (2.43)$$
$$\sigma_{31}n_1^{(2)} + \sigma_{32}n_2^{(2)} + (\sigma_{33} - \sigma_{(2)})n_3^{(2)} = 0$$

Because the stress tensor is real and symmetric, the principal stress values are also real.

When referred to principal stress directions, the stress matrix $[\sigma_{ij}]$ is diagonal,

$$[\sigma_{ij}] \equiv \begin{bmatrix} \sigma_{(1)} & 0 & 0 \\ 0 & \sigma_{(2)} & 0 \\ 0 & 0 & \sigma_{(3)} \end{bmatrix} \quad \text{or} \quad [\sigma_{ij}] = \begin{bmatrix} \sigma_{\mathrm{I}} & 0 & 0 \\ 0 & \sigma_{\mathrm{II}} & 0 \\ 0 & 0 & \sigma_{\mathrm{III}} \end{bmatrix} \qquad (2.44)$$

in the second form of which Roman numeral subscripts are used to show that the principal stresses are ordered, i.e. $\sigma_{\mathrm{I}} > \sigma_{\mathrm{II}} > \sigma_{\mathrm{III}}$. Since the principal stress directions are coincident with the principal axes of Cauchy's stress quadric, the principal stress values include both the maximum and minimum normal stress components at a point.

In a *principal stress space*, i.e. a space whose axes are in the principal stress directions and whose coordinate unit of measure is stress $(t_1^{(\hat{n})}, t_2^{(\hat{n})}, t_3^{(\hat{n})})$ as shown in Fig. 2-12, the arbitrary stress vector $t_i^{(\hat{n})}$ has components

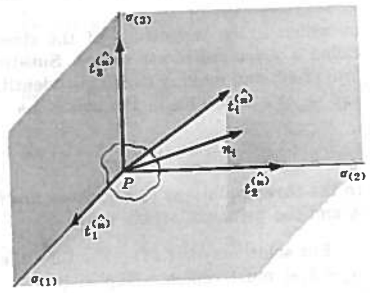

Fig. 2-12

$$t_1^{(\hat{n})} = \sigma_{(1)}n_1, \qquad t_2^{(\hat{n})} = \sigma_{(2)}n_2, \qquad t_3^{(\hat{n})} = \sigma_{(3)}n_3 \qquad (2.45)$$

according to (*2.12*). But inasmuch as $(n_1)^2 + (n_2)^2 + (n_3)^2 = 1$ for the unit vector n_i, (*2.45*) requires the stress vector $t_i^{(\hat{n})}$ to satisfy the equation

$$\frac{(t_1^{(\hat{n})})^2}{(\sigma_{(1)})^2} + \frac{(t_2^{(\hat{n})})^2}{(\sigma_{(2)})^2} + \frac{(t_3^{(\hat{n})})^2}{(\sigma_{(3)})^2} = 1 \qquad (2.46)$$

in stress space. This equation is an ellipsoid known as the *Lamé stress ellipsoid*.

2.11 MAXIMUM AND MINIMUM SHEAR STRESS VALUES

If the stress vector $t_i^{(\hat{n})}$ is resolved into orthogonal components normal and tangential to the surface element dS upon which it acts, the magnitude of the normal component may be determined from (*2.33*) and the magnitude of the tangential or *shearing component* is given by

$$\sigma_S^2 = t_i^{(\hat{n})}t_i^{(\hat{n})} - \sigma_N^2 \qquad (2.47)$$

This resolution is shown in Fig. 2-13 where the axes are chosen in the principal stress directions and it is assumed the principal stresses are ordered according to $\sigma_{\mathrm{I}} > \sigma_{\mathrm{II}} > \sigma_{\mathrm{III}}$. Hence from (*2.12*), the components of $t_i^{(\hat{n})}$ are

$$t_1^{(\hat{n})} = \sigma_{\mathrm{I}}n_1$$
$$t_2^{(\hat{n})} = \sigma_{\mathrm{II}}n_2 \qquad (2.48)$$
$$t_3^{(\hat{n})} = \sigma_{\mathrm{III}}n_3$$

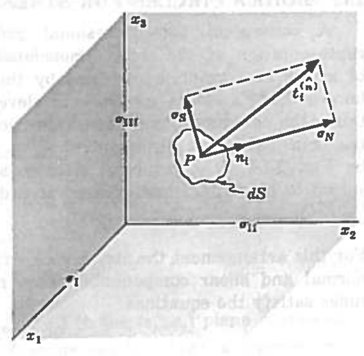

Fig. 2-13

and from (*2.33*), the normal component magnitude is

$$\sigma_N = \sigma_{\mathrm{I}}n_1^2 + \sigma_{\mathrm{II}}n_2^2 + \sigma_{\mathrm{III}}n_3^2 \qquad (2.49)$$

Substituting (*2.48*) and (*2.49*) into (*2.47*), the squared magnitude of the shear stress as a function of the direction cosines n_i is given by

$$\sigma_S^2 = \sigma_{\mathrm{I}}^2n_1^2 + \sigma_{\mathrm{II}}^2n_2^2 + \sigma_{\mathrm{III}}^2n_3^2 - (\sigma_{\mathrm{I}}n_1^2 + \sigma_{\mathrm{II}}n_2^2 + \sigma_{\mathrm{III}}n_3^2)^2 \qquad (2.50)$$

The maximum and minimum values of σ_S may be obtained from (*2.50*) by the method of *Lagrangian multipliers*. The procedure is to construct the function

$$F = \sigma_S^2 - \lambda n_i n_i \qquad (2.51)$$

in which the scalar λ is called a Lagrangian multiplier. Equation (*2.51*) is clearly a function of the direction cosines n_i, so that the conditions for stationary (maximum or minimum) values of F are given by $\partial F/\partial n_i = 0$. Setting these partials equal to zero yields the equations

$$n_1\{\sigma_{\mathrm{I}}^2 - 2\sigma_{\mathrm{I}}(\sigma_{\mathrm{I}}n_1^2 + \sigma_{\mathrm{II}}n_2^2 + \sigma_{\mathrm{III}}n_3^2) + \lambda\} = 0 \qquad (2.52a)$$
$$n_2\{\sigma_{\mathrm{II}}^2 - 2\sigma_{\mathrm{II}}(\sigma_{\mathrm{I}}n_1^2 + \sigma_{\mathrm{II}}n_2^2 + \sigma_{\mathrm{III}}n_3^2) + \lambda\} = 0 \qquad (2.52b)$$
$$n_3\{\sigma_{\mathrm{III}}^2 - 2\sigma_{\mathrm{III}}(\sigma_{\mathrm{I}}n_1^2 + \sigma_{\mathrm{II}}n_2^2 + \sigma_{\mathrm{III}}n_3^2) + \lambda\} = 0 \qquad (2.52c)$$

which, together with the condition $n_i n_i = 1$, may be solved for λ and the direction cosines n_1, n_2, n_3, conjugate to the extremum values of shear stress.

One set of solutions to (*2.52*), and the associated shear stresses from (*2.50*), are

$$n_1 = \pm 1, \quad n_2 = 0, \quad n_3 = 0; \quad \text{for which } \sigma_S = 0 \qquad (2.53a)$$
$$n_1 = 0, \quad n_2 = \pm 1, \quad n_3 = 0; \quad \text{for which } \sigma_S = 0 \qquad (2.53b)$$
$$n_1 = 0, \quad n_2 = 0, \quad n_3 = \pm 1; \quad \text{for which } \sigma_S = 0 \qquad (2.53c)$$

The shear stress values in (*2.53*) are obviously minimum values. Furthermore, since (*2.35*) indicates that shear components vanish on principal planes, the directions given by (*2.53*) are recognized as principal stress directions.

A second set of solutions to (*2.52*) may be verified to be given by

$$n_1 = 0, \quad n_2 = \pm 1/\sqrt{2}, \quad n_3 = \pm 1/\sqrt{2}; \quad \text{for which } \sigma_S = (\sigma_{\mathrm{II}} - \sigma_{\mathrm{III}})/2 \qquad (2.54a)$$
$$n_1 = \pm 1/\sqrt{2}, \quad n_2 = 0, \quad n_3 = \pm 1/\sqrt{2}; \quad \text{for which } \sigma_S = (\sigma_{\mathrm{III}} - \sigma_{\mathrm{I}})/2 \qquad (2.54b)$$
$$n_1 = \pm 1/\sqrt{2}, \quad n_2 = \pm 1/\sqrt{2}, \quad n_3 = 0; \quad \text{for which } \sigma_S = (\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})/2 \qquad (2.54c)$$

Equation (*2.54b*) gives the maximum shear stress value, which is equal to half the difference of the largest and smallest principal stresses. Also from (*2.54b*), the maximum shear stress component acts in the plane which bisects the right angle between the directions of the maximum and minimum principal stresses.

2.12 MOHR'S CIRCLES FOR STRESS

A convenient two-dimensional graphical representation of the three-dimensional state of stress at a point is provided by the well-known *Mohr's stress circles*. In developing these, the coordinate axes are again chosen in the principal stress directions at P as shown by Fig. 2-14. The principal stresses are assumed to be distinct and ordered according to

$$\sigma_I > \sigma_{II} > \sigma_{III} \tag{2.55}$$

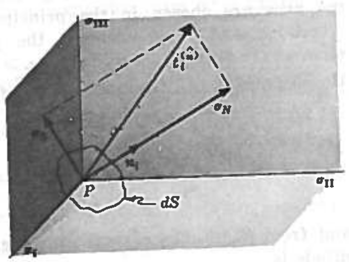

Fig. 2-14

For this arrangement the stress vector $t_i^{(\hat{n})}$ has normal and shear components whose magnitudes satisfy the equations

$$\sigma_N = \sigma_I n_1^2 + \sigma_{II} n_2^2 + \sigma_{III} n_3^2 \tag{2.56}$$

$$\sigma_N^2 + \sigma_S^2 = \sigma_I^2 n_1^2 + \sigma_{II}^2 n_2^2 + \sigma_{III}^2 n_3^2 \tag{2.57}$$

Combining these two expressions with the identity $n_i n_i = 1$ and solving for the direction cosines n_i, results in the equations

$$(n_1)^2 = \frac{(\sigma_N - \sigma_{II})(\sigma_N - \sigma_{III}) + (\sigma_S)^2}{(\sigma_I - \sigma_{II})(\sigma_I - \sigma_{III})} \tag{2.58a}$$

$$(n_2)^2 = \frac{(\sigma_N - \sigma_{III})(\sigma_N - \sigma_I) + (\sigma_S)^2}{(\sigma_{II} - \sigma_{III})(\sigma_{II} - \sigma_I)} \tag{2.58b}$$

$$(n_3)^2 = \frac{(\sigma_N - \sigma_I)(\sigma_N - \sigma_{II}) + (\sigma_S)^2}{(\sigma_{III} - \sigma_I)(\sigma_{III} - \sigma_{II})} \tag{2.58c}$$

These equations serve as the basis for Mohr's stress circles, shown in the "stress plane" of Fig. 2-15, for which the σ_N axis is the abscissa, and the σ_S axis is the ordinate.

In (2.58a), since $\sigma_I - \sigma_{II} > 0$ and $\sigma_I - \sigma_{III} > 0$ from (2.55), and since $(n_1)^2$ is non-negative, the numerator of the right-hand side satisfies the relationship

$$(\sigma_N - \sigma_{II})(\sigma_N - \sigma_{III}) + (\sigma_S)^2 \geqq 0 \tag{2.59}$$

which represents stress points in the (σ_N, σ_S) plane that are *on* or *exterior* to the circle

$$[\sigma_N - (\sigma_{II} + \sigma_{III})/2]^2 + (\sigma_S)^2 = [(\sigma_{II} - \sigma_{III})/2]^2 \tag{2.60}$$

In Fig. 2-15, this circle is labeled C_1.

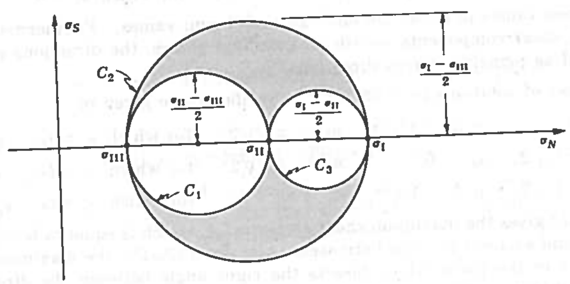

Fig. 2-15

Similarly, for (2.58b), since $\sigma_{II} - \sigma_{III} > 0$ and $\sigma_{II} - \sigma_I < 0$ from (2.55), and since $(n_2)^2$ is non-negative, the right hand numerator satisfies

$$(\sigma_N - \sigma_{III})(\sigma_N - \sigma_I) + (\sigma_S)^2 \leqq 0 \tag{2.61}$$

which represents points *on* or *interior* to the circle

$$[\sigma_N - (\sigma_{III} + \sigma_I)/2]^2 + (\sigma_S)^2 = [(\sigma_{III} - \sigma_I)/2]^2 \tag{2.62}$$

labeled C_2 in Fig. 2-15. Finally, for (2.58c), since $\sigma_{III} - \sigma_I < 0$ and $\sigma_{III} - \sigma_{II} < 0$ from (2.55), and since $(n_3)^2$ is non-negative,

$$(\sigma_N - \sigma_I)(\sigma_N - \sigma_{II}) + (\sigma_S)^2 \geqq 0 \tag{2.63}$$

which represents points *on* or *exterior* to the circle

$$[\sigma_N - (\sigma_I + \sigma_{II})/2]^2 + (\sigma_S)^2 = [(\sigma_I - \sigma_{II})/2]^2 \tag{2.64}$$

labeled C_3 in Fig. 2-15.

Since each "stress point" (pair of values of σ_N and σ_S) in the (σ_N, σ_S) plane represents a particular stress vector $t_i^{(\hat{n})}$, the state of stress at P expressed by (2.58) is represented in Fig. 2-15 as the shaded area bounded by the Mohr's stress circles. The diagram confirms a maximum shear stress of $(\sigma_I - \sigma_{III})/2$ as was determined analytically in Section 2.11. Frequently, because the sign of the shear stress is not of critical importance, only the top half of this symmetrical diagram is drawn.

The relationship between Mohr's stress diagram and the physical state of stress may be established through consideration of Fig. 2-16, which shows the first octant of a sphere of the continuum centered at point P. The normal n_i at the arbitrary point Q of the spherical surface ABC simulates the normal to the surface element dS at point P. Because of the symmetry properties of the stress tensor and the fact that principal stress axes are used in Fig. 2-16, the state of stress at P is completely represented through the totality of locations Q can occupy on the surface ABC. In the figure, circle arcs KD, GE and FH designate locations for Q along which one direction cosine of n_i has a constant value. Specifically,

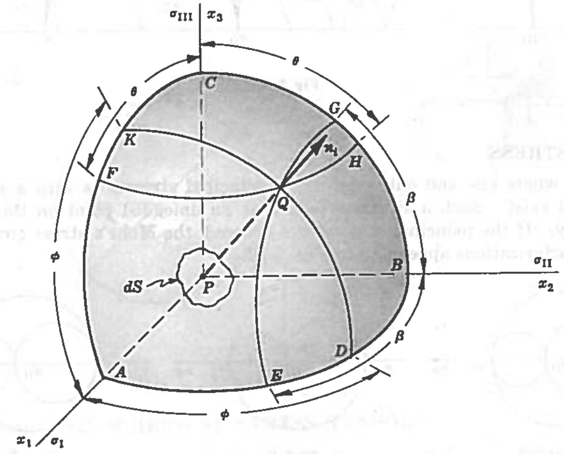

Fig. 2-16

$$n_1 = \cos\phi \text{ on } KD, \quad n_2 = \cos\beta \text{ on } GE, \quad n_3 = \cos\theta \text{ on } FH$$

and, on the bounding circle arcs BC, CA and AB,

$$n_1 = \cos\pi/2 = 0 \text{ on } BC, \quad n_2 = \cos\pi/2 = 0 \text{ on } CA, \quad n_3 = \cos\pi/2 = 0 \text{ on } AB$$

According to the first of these and the equation (*2.58a*), stress vectors for Q located on BC will have components given by stress points on the circle C_1 in Fig. 2-15. Likewise, CA in Fig. 2-16 corresponds to the circle C_2, and AB to the circle C_3 in Fig. 2-15.

The stress vector components σ_N and σ_S for an arbitrary location of Q may be determined by the construction shown in Fig. 2-17. Thus point e may be located on C_3 by drawing the radial line from the center of C_3 at the angle 2β. Note that angles in the physical space of Fig. 2-16 are doubled in the stress space of Fig. 2-17 (arc AB subtends $90°$ in Fig. 2-16 whereas the conjugate stress points σ_I and σ_{II} are $180°$ apart on C_3). In the same way, points g, h and f are located in Fig. 2-17 and the appropriate pairs joined by circle arcs having their centers on the σ_N axis. The intersection of circle arcs ge and hf represents the components σ_N and σ_S of the stress vector $t_i^{(\hat{n})}$ on the plane having the normal direction n_i at Q in Fig. 2-16.

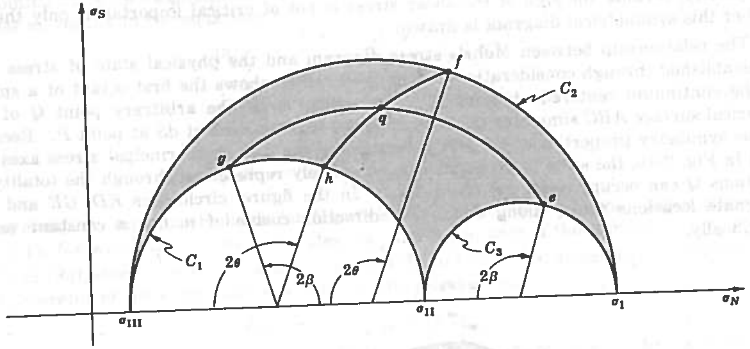

Fig. 2-17

2.13 PLANE STRESS

In the case where one and only one of the principal stresses is zero a state of *plane stress* is said to exist. Such a situation occurs at an unloaded point on the free surface bounding a body. If the principal stresses are ordered, the Mohr's stress circles will have one of the characterizations appearing in Fig. 2-18.

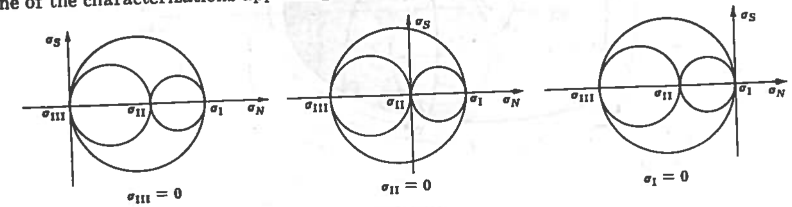

Fig. 2-18

If the principal stresses are not ordered and the direction of the zero principal stress is taken as the x_3 direction, the state of stress is termed plane stress parallel to the x_1x_2 plane. For arbitrary choice of orientation of the orthogonal axes x_1 and x_2 in this case, the stress matrix has the form

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.65}$$

The stress quadric for this plane stress is a cylinder with its base lying in the x_1x_2 plane and having the equation

$$\sigma_{11}x_1^2 + 2\sigma_{12}x_1x_2 + \sigma_{22}x_2^2 = \pm k^2 \tag{2.66}$$

Frequently in elementary books on Strength of Materials a state of plane stress is represented by a single Mohr's circle. As seen from Fig. 2-18 this representation is necessarily incomplete since all three circles are required to show the complete stress picture. In particular, the maximum shear stress value at a point will not be given if the single circle presented happens to be one of the inner circles of Fig. 2-18. A single circle Mohr's diagram is able, however, to display the stress points for all those planes at the point P which include the zero principal stress axis. For such planes, if the coordinate axes are chosen in accordance with the stress representation given in (*2.65*), the single plane stress Mohr's circle has the equation

$$[\sigma_N - (\sigma_{11} + \sigma_{22})/2]^2 + (\sigma_S)^2 = [(\sigma_{11} - \sigma_{22})/2]^2 + (\sigma_{12})^2 \tag{2.67}$$

The essential features in the construction of this circle are illustrated in Fig. 2-19. The circle is drawn by locating the center C at $\sigma_N = (\sigma_{11} + \sigma_{22})/2$ and using the radius $R = \sqrt{[(\sigma_{11} - \sigma_{22})/2]^2 + (\sigma_{12})^2}$ given in (*2.67*). Point A on the circle represents the stress state on the surface element whose normal is n_1 (the right-hand face of the rectangular parallelepiped shown in Fig. 2-19). Point B on the circle represents the stress state on the top surface of the parallelepiped with normal n_2. Principal stress points σ_I and σ_{II} are so labeled, and points E and D on the circle are points of maximum shear stress value.

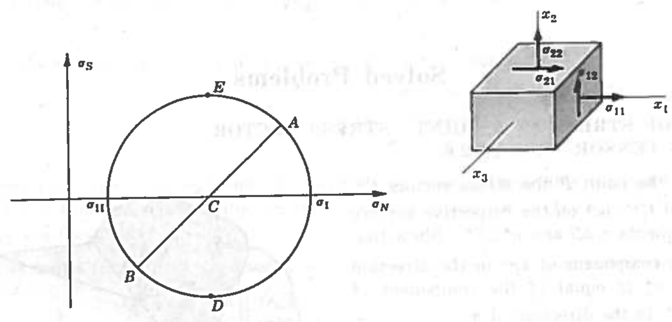

Fig. 2-19

2.14 DEVIATOR AND SPHERICAL STRESS TENSORS

It is very often useful to split the stress tensor σ_{ij} into two component tensors, one of which (the *spherical* or *hydrostatic stress tensor*) has the form

$$\Sigma_M = \sigma_M \mathbf{I} = \begin{pmatrix} \sigma_M & 0 & 0 \\ 0 & \sigma_M & 0 \\ 0 & 0 & \sigma_M \end{pmatrix} \quad (2.68)$$

where $\sigma_M = -p = \sigma_{kk}/3$ is the mean normal stress, and the second (the *deviator stress tensor*) has the form

$$\Sigma_D = \begin{pmatrix} \sigma_{11}-\sigma_M & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma_M & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma_M \end{pmatrix} \equiv \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix} \quad (2.69)$$

This decomposition is expressed by the equations

$$\sigma_{ij} = \delta_{ij}\sigma_{kk}/3 + s_{ij} \quad \text{or} \quad \Sigma = \sigma_M \mathbf{I} + \Sigma_D \quad (2.70)$$

The principal directions of the deviator stress tensor s_{ij} are the same as those of the stress tensor σ_{ij}. Thus *principal deviator stress* values are

$$s_{(k)} = \sigma_{(k)} - \sigma_M \quad (2.71)$$

The characteristic equation for the deviator stress tensor, comparable to (2.38) for the stress tensor, is the cubic

$$s^3 + \mathrm{II}_{\Sigma_D} s - \mathrm{III}_{\Sigma_D} = 0 \quad \text{or} \quad s^3 + (s_I s_{II} + s_{II} s_{III} + s_{III} s_I)s - s_I s_{II} s_{III} = 0 \quad (2.72)$$

It is easily shown that the first invariant of the deviator stress tensor I_{Σ_D} is identically zero, which accounts for its absence in (2.72).

Solved Problems

STATE OF STRESS AT A POINT. STRESS VECTOR. STRESS TENSOR (Sec. 2.1-2.6)

2.1. At the point P the stress vectors $t_i^{(\hat{n})}$ and $t_i^{(\hat{n}^*)}$ act on the respective surface elements $n_i \Delta S$ and $n_i^* \Delta S^*$. Show that the component of $t_i^{(\hat{n})}$ in the direction of n_i^* is equal to the component of $t_i^{(\hat{n}^*)}$ in the direction of n_i.

It is required to show that

$$t_i^{(\hat{n}^*)} n_i = t_i^{(\hat{n})} n_i^*$$

From (2.12) $t_i^{(\hat{n}^*)} n_i = \sigma_{ji} n_j^* n_i$, and by (2.22) $\sigma_{ji} = \sigma_{ij}$, so that

$$\sigma_{ji} n_j^* n_i = (\sigma_{ij} n_i) n_j^* = t_j^{(\hat{n})} n_j^*$$

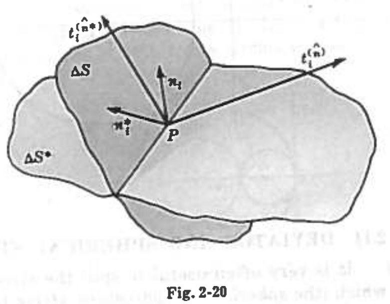

Fig. 2-20

2.2. The stress tensor values at a point P are given by the array

$$\Sigma = \begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{pmatrix}$$

Determine the traction (stress) vector on the plane at P whose unit normal is $\hat{\mathbf{n}} = (2/3)\hat{\mathbf{e}}_1 - (2/3)\hat{\mathbf{e}}_2 + (1/3)\hat{\mathbf{e}}_3$.

From (2.12), $\mathbf{t}^{(\hat{n})} = \hat{\mathbf{n}} \cdot \Sigma$. The multiplication is best carried out in the matrix form of (2.13):

$$[t_1^{(\hat{n})}, t_2^{(\hat{n})}, t_3^{(\hat{n})}] = [2/3, -2/3, 1/3] \begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix} = \left[\frac{14}{3} - \frac{2}{3}, \frac{-10}{3}, \frac{-4}{3} + \frac{4}{3}\right]$$

Thus $\mathbf{t}^{(\hat{n})} = 4\hat{\mathbf{e}}_1 - \frac{10}{3}\hat{\mathbf{e}}_2$.

2.3. For the traction vector of Problem 2.2, determine (*a*) the component perpendicular to the plane, (*b*) the magnitude of $t_i^{(\hat{n})}$, (*c*) the angle between $t_i^{(\hat{n})}$ and $\hat{\mathbf{n}}$.

(*a*) $t_i^{(\hat{n})} \cdot \hat{\mathbf{n}} = (4\hat{\mathbf{e}}_1 - \frac{10}{3}\hat{\mathbf{e}}_2) \cdot (\frac{2}{3}\hat{\mathbf{e}}_1 - \frac{2}{3}\hat{\mathbf{e}}_2 + \frac{1}{3}\hat{\mathbf{e}}_3) = 44/9$

(*b*) $|t_i^{(\hat{n})}| = \sqrt{16 + 100/9} = 5.2$

(*c*) Since $t_i^{(\hat{n})} \cdot \hat{\mathbf{n}} = |t_i^{(\hat{n})}| \cos\theta$, $\cos\theta = (44/9)/5.2 = 0.94$ and $\theta = 20°$.

2.4. The stress vectors acting on the three coordinate planes are given by $t_i^{(\hat{e}_1)}$, $t_i^{(\hat{e}_2)}$ and $t_i^{(\hat{e}_3)}$. Show that the sum of the squares of the magnitudes of these vectors is independent of the orientation of the coordinate planes.

Let S be the sum in question. Then

$$S = t_i^{(\hat{e}_1)} t_i^{(\hat{e}_1)} + t_i^{(\hat{e}_2)} t_i^{(\hat{e}_2)} + t_i^{(\hat{e}_3)} t_i^{(\hat{e}_3)}$$

which from (2.7) becomes $S = \sigma_{1i}\sigma_{1i} + \sigma_{2i}\sigma_{2i} + \sigma_{3i}\sigma_{3i} = \sigma_{ji}\sigma_{ji}$, an invariant.

2.5. The state of stress at a point is given by the stress tensor

$$\sigma_{ij} = \begin{pmatrix} \sigma & a\sigma & b\sigma \\ a\sigma & \sigma & c\sigma \\ b\sigma & c\sigma & \sigma \end{pmatrix}$$

where a, b, c are constants and σ is some stress value. Determine the constants a, b and c so that the stress vector on the *octahedral* plane $(\hat{\mathbf{n}} = (1/\sqrt{3})\hat{\mathbf{e}}_1 + (1/\sqrt{3})\hat{\mathbf{e}}_2 + (1/\sqrt{3})\hat{\mathbf{e}}_3)$ vanishes.

In matrix form, $t_i^{(\hat{n})} = \sigma_{ij} n_j$ must be zero for the given stress tensor and normal vector.

$$\begin{bmatrix} \sigma & a\sigma & b\sigma \\ a\sigma & \sigma & c\sigma \\ b\sigma & c\sigma & \sigma \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{hence} \quad \begin{aligned} a + b &= -1 \\ a + c &= -1 \\ b + c &= -1 \end{aligned}$$

Solving these equations, $a = b = c = -1/2$. Therefore the solution tensor is

$$\sigma_{ij} = \begin{pmatrix} \sigma & -\sigma/2 & -\sigma/2 \\ -\sigma/2 & \sigma & -\sigma/2 \\ -\sigma/2 & -\sigma/2 & \sigma \end{pmatrix}$$