Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact during exam:
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FINAL EXAM IN TMA4220

NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS USING THE FINITE ELEMENT METHOD

Saturday December 18, 2010
Time: 09:00–13:00

Permitted aids: Approved calculator.
All printed and hand written aids.

Problem 1 Consider the two-dimensional Poisson equation

$$-\nabla^2 u = f \quad \text{in} \quad \Omega \tag{1}$$

with the boundary conditions

$$\begin{aligned} u &= 0 \quad \text{on} \quad \partial\Omega_D & (2) \\ H(u) &= \bar{t} \text{ on } \partial\Omega_N. & (3) \end{aligned}$$

where u is the unknown solution, f the applied loading, $\bar{t}$ are prescribed Neumann boundary conditions for the given Poisson problem defined on the polygonal domain $\Omega \in R^2$. The differential operator $H(u)$ is the so called "Neumann operator".

a) Use Galerkin's method and establish the weak formulation corresponding to the problem (1)-(3) on the form: Find $u \in X$ such that

$$a(u, v) = l(v) \qquad \forall v \in X \tag{4}$$

In particular, identify X, a and l for this problem. Identify also the expression for the "Neumann operator" $H(u)$ related to the Neumann boundary conditions.

b) Assume that we choose the finite element method to solve (4) numerically. Formulate the corresponding finite element variational formulation.

In Figure 1 we have displayed a domain Ω discretized with 6 linear triangles. We assume homogeneous Dirichlet and Neumann boundary conditions along $\partial\Omega_D$ and $\partial\Omega_N$, respectively. Here $\partial\Omega_N$ is the union of the halfopen interval $(-1, 0]$ along the x-axis and the halfopen interval $[0, 1)$ along the y-axis, whereas $\partial\Omega_D = \partial\Omega \setminus \partial\Omega_N$. Let the applied loading f be a constant defined on the whole domain Ω.

c) Compute the terms $a(\phi_1, \phi_1)$ and $l(\phi_1)$, where ϕ_1 is the basis function connected to node 1 in Figure 1.

d) Find $u_h \in X_h$, i.e. solve the finite element system for the problem displayed in Figure 1.

e) If you were free to choose how to discretise the given domain Ω with 6 linear triangles would you use the same finite element mesh as shown in Figure 1 or would you change it. Justify your answer.

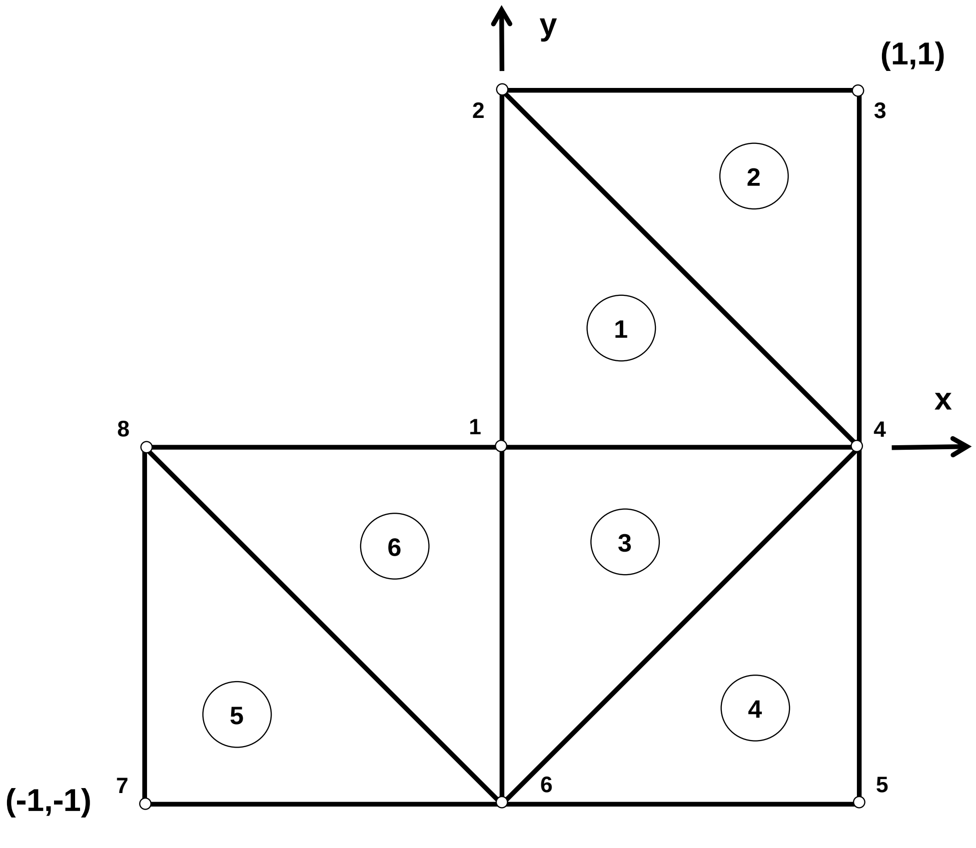

Figure 1: The domain Ω discretized with 6 linear triangles and 8 nodes.

Problem 2

a) What characterize a finite element?

b) Define a compatible triangular Lagrange type finite element with 4 nodes as displayed in Figure 2, where the interior node is in the barycenter. Use barycentric coordinates and define the four element nodal basis functions.

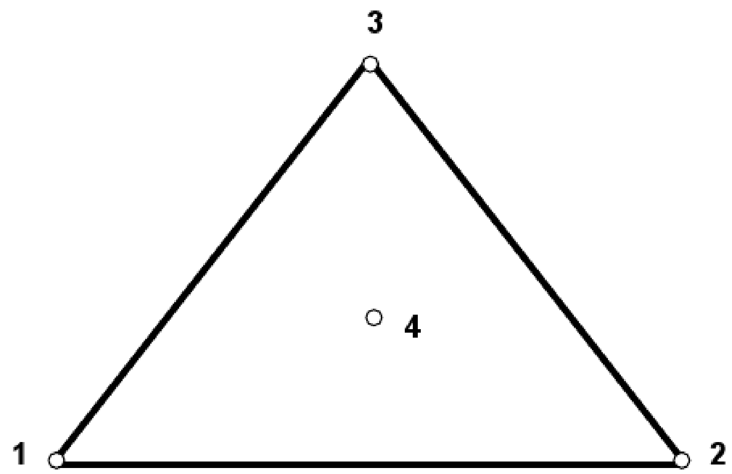

Figure 2: Compatible triangle with 4 nodes.

c) Define a compatible biquartic (i.e. 4^{rd} order) quadrilateral Lagrange type finite element, i.e. show all the nodal locations and the corresponding element nodal basis functions for one vertex, one edge node and one interior node (i.e. three in total).

d) Define "affine coordinate mapping". Which property inherent in affine coordinate mappings is consider to be of special interest related to the finite element method.

e) Define the nodal basis functions for the 4 node "transfer element" shown in Figure 3, such that it is compatible with two neighboring standard 3 node linear Lagrange triangular element along the edge with the midpoint node, and compatible with a neighboring 3 node linear Lagrange triangular element along the two other edges.

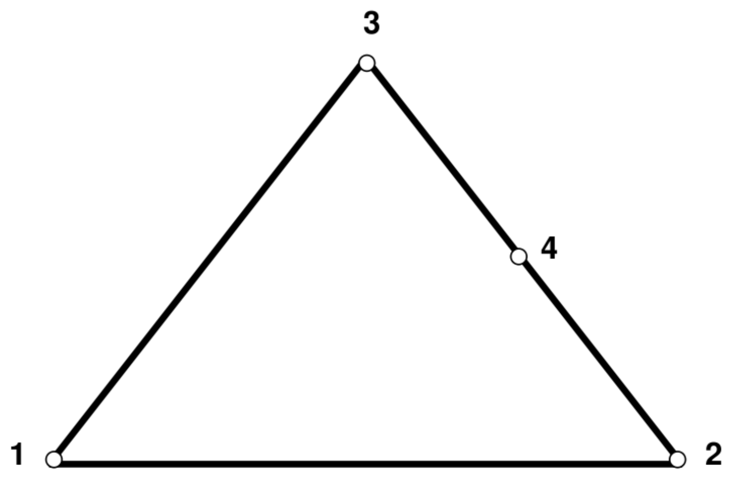

Figure 3: "Transfer element" with 4 nodes.

Problem 3 Consider the unsteady heat equation:

$$\frac{\partial u}{\partial t} - \kappa \nabla^2 u = f \quad \text{in} \quad \Omega \times [0, T], \tag{5}$$

$$u = \bar{u} \quad \text{on} \quad \partial\Omega, \tag{6}$$

$$u(\underline{x}, t = 0) = u_0(\underline{x}), \tag{7}$$

where κ is a constant, positive diffusion coefficient, u is the unknown time-dependent temperature, and $\bar{u}$ is prescribed Dirichlet boundary conditions.

a) Derive a weak formulation of the problem (5)-(7) where you identify the appropriate functions spaces and forms.

b) Assume that we choose the finite element method to solve this problem numerically. Formulate the corresponding semidiscrete finite element variational problem.

c) Explain how one may solve the semidiscrete finite element variational problem.