Norwegian University of Science and Technology
Department of Mathematical Sciences

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Exam in TMA4220
Numerical Solution of Partial Differential Equations Using Element Methods

Wednesday December 5, 2012
Time: 15.00 – 19.00

Auxiliary materials: Simple calculator (Hewlett Packard HP30S or Citizen SR-270X)
All printed and hand written material.

Deadline for the grading: 21.12.2012.

Problem 1

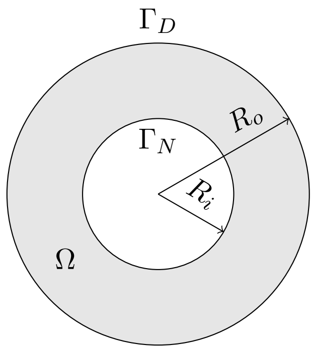
Given the Poisson equation

$$-\Delta u = f \quad \text{in} \quad \Omega \tag{1}$$

with boundary conditions

$$u = 0 \text{ on } \Gamma_D, \qquad \frac{\partial u}{\partial n} = q \text{ on } \Gamma_N.$$

where Ω is the domain between a circle of radius R_i and one of radius R_o, Γ_D is the outer boundary and Γ_N the inner.



a) Establish the weak formulation

$$\text{find } u \in V \text{ such that} \qquad a(u,v) = F(v), \qquad \forall v \in V. \tag{2}$$

That means, identify V and find expressions for a and F,

Solution:
Multiply by a test function v on both sizes, integrate over Ω and use Greens Theorem:

$$\int_\Omega \nabla u \nabla v d\Omega - \int_{\Gamma_D} \frac{\partial u}{\partial n} v d\gamma + \int_{\Gamma_N} \frac{\partial u}{\partial n} v\gamma = \int_\Omega f v d\Omega.$$

Now, for the integrals to exist, we require $v, u \in H^1(\Omega)$. Further, since we now nothing about $\partial u/\partial n$ on Γ_D we let $v = 0$ on that boundary. Which, by pure chance is the same boundary condition as imposed on u. So

$$V = \{v \in H^1(\Omega) \,:\, v|_{\Gamma_D} = 0\} = H^1_{\Gamma_D}(\Omega),$$

$$a(u,v) = \int_\Omega \nabla u \nabla v d\Omega, \qquad F(v) = \int_\Omega f v d\Omega + \int_{\Gamma_N} q v d\gamma.$$

In the following, let $f = -4$, $q = 0.5$, $R_i = 0.5$ and $R_o = 1$. Since f and q are both constant, we note that the solution u only depend on the distance from the center r, and the problem is reduced to a one-dimensional case.

b) Show that the bilinear form $a(u,v)$ in the weak formulation (2) now is

$$a(u,v) = \int_{0.5}^{1} r \, \frac{\partial u}{\partial r}\frac{\partial v}{\partial r}\, dr, \qquad V = \{v \in H^1(0.5, 1) \,:\, v(1) = 0\}.$$

Find also an expression for $F(v)$ in this case.

Hint: Use polar coordinates, see the appendix at the end of the set.

Solution:
Using polar coordinates, you get

$$a(u,v) = \int_{R_i}^{R_o}\int_0^{2\pi} \frac{\partial u}{\partial r}\frac{\partial v}{\partial r} d\theta \, dr = 2\pi \int_{R_i}^{R_o} r \frac{\partial u}{\partial r}\frac{\partial v}{\partial r} dr.$$

Similar,

$$F(v) = 2\pi \cdot f \cdot \int_{R_i}^{R_o} r v dr - 2\pi R_i q v(R_i)$$

So, dividing by 2π on both sides gives the expression for $a(u,v)$ and

$$F(v) = -4 \int_{0.5}^{1} r v dr - 0.25\, v(0.5).$$

c) Find the exact solution $u(r)$ of this problem.

Solution:
First: Find the strong form of the problem. By using partial integration we get

$$a(u,v) = -\int_{R_i}^{R_o} \frac{\partial}{\partial r}(r \frac{\partial u}{\partial r}) v + r\, \frac{\partial u}{\partial r} v \Big|_{R_i}^{R_o}$$

so that

$$a(u,v) - F(v) = -\int_{R_i}^{R_o} \left(\frac{\partial}{\partial r}(r \frac{\partial u}{\partial r}) + f r \right) v dr + r(\frac{\partial u}{\partial r} + q) v \Big|_{R_i}^{R_o} = 0$$

which is true for all $v \in V$ if

$$-\frac{\partial}{\partial r}(r \frac{\partial u}{\partial r}) = f r, \qquad \frac{\partial u}{\partial r}(R_i) = -q, \qquad u(R_o) = 0.$$

Or you could simply use (1) in polar coordinates, and you have the equations directly. Inserting the given values gives

$$\frac{\partial}{\partial r}(r \frac{\partial u}{\partial r}) = 4r, \qquad \frac{\partial u}{\partial r}(\frac{1}{2}) = -\frac{1}{2}, \qquad u(1) = 0.$$

Integrating twice, and using the boundary conditions gives

$$u(r) = r^2 - 1 - \frac{3}{4}\ln(r).$$

We will now like to find an approximation to the solution of the one-dimensional problem by use of the finite element method with linear, nodal basis functions on a uniform grid, that is $V_h = X_h^1$, and $h = 0.5/M$.

d) Find the elemental stiffness matrix and the elemental load vector for the element $K = [r_{k-1}, r_k]$, where $r_k = 0.5 + h \cdot k,\ \ k = 1, 2, \cdots, M$.

Solution:
The linear basis functions are

$$\varphi_{k-1}^K(r) = \frac{r_k - r}{h}, \qquad \varphi_k^K(r) = \frac{r - r_{k-1}}{h}.$$

Thus

$$a_{k-1,k-1}^K = \int_{r_{k-1}}^{r_k} r\frac{1}{h}\frac{1}{h}dr = \frac{r_k^2 - r_{k-1}^2}{2h^2} = \frac{r_k + r_{k-1}}{2h}, \qquad a_{k-1,k}^K = \int_{r_{k-1}}^{r_k} r\frac{1}{h}(-\frac{1}{h})dr = -\frac{r_k + r_{k-1}}{2h}, \qquad \textit{etc},$$

$$b_{k-1}^K = -4\int_{r_{k-1}}^{r_k} r\frac{r_k - r}{h}dr = -\frac{2}{3}(2r_{k-1}+r_k)h, b_k^K = -4\int_{r_{k-1}}^{r_k} r\frac{r - r_{k-1}}{h}dr = -\frac{2}{3}(r_{k-1}+2r_k)h.$$

So

$$A^K = \frac{r_k + r_{k-1}}{2h} \cdot \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad b^K = -\frac{2h}{3} \cdot \begin{pmatrix} 2r_{k-1} + r_k \\ r_{k-1} + 2r_k \end{pmatrix}.$$

e) The finite element method can be formulated as

$$A_h \mathbf{u}_h = \mathbf{b}_h$$

where $\mathbf{u}_h = (u_0, u_1, \cdots, u_{M-1})^T$ where $u_k \approx u(r_k)$. Show that A_h is the symmetric tridiagonal matrix

$$A_h = \begin{pmatrix} \gamma_0 & \beta_0 & & & \\ \beta_0 & \gamma_1 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{M-2} \\ & & & \beta_{M-2} & \gamma_{M-1} \end{pmatrix}$$

and find for β_k and γ_k (only for $k \neq 0$ and $k \neq M-1$.).
Solution:
Using a standard assembly, we get (for $k > 0$) that

$$\gamma_k = a_{k,k}^K + a_{k,k}^{K+1} = \frac{r_{k-1} + 2r_k + r_{k-1}}{2h} = \frac{2r_k}{h}.$$

$$\beta_k = a_{k,k+1}^{K+1} = a_{k+1,k}^{K+1} = -\frac{r_k + r_{k+1}}{2h}.$$

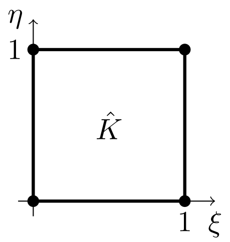
The stiffness matrix is symmetric since the bilinear form $a(u,v)$ is symmetric.

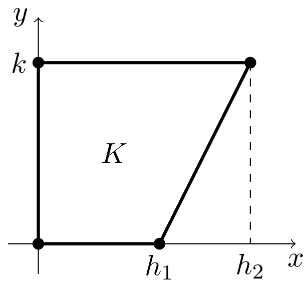
Problem 2

a) Given a quadratic reference finite element $\hat{K}$, with nodes in the corners. Write down the four bilinear nodal basis functions $\hat{\varphi}_{\hat{\alpha}}(\xi, \eta)$ for this element. The index $\hat{\alpha}$ refers to the nodes.

Solution:
With $\hat{A} = (0,0)$, $\hat{B} = (1,0)$, $\hat{C} = (1,0)$ *and* $\hat{D} = (1,1)$ *we get the bilinear nodal basis functions:*

$$\hat{\varphi}_A(\xi,\eta) = (1-\xi)(1-\eta) \qquad \hat{\varphi}_B(\xi,\eta) = \xi(1-\eta)$$
$$\hat{\varphi}_C(\xi,\eta) = (1-\xi)\eta \qquad \hat{\varphi}_D(\xi,\eta) = \xi\eta$$





b) Find a bilinear mapping $x(\xi,\eta), y(\xi,\eta)$ mapping each node from the quadratic reference element to the corresponding node of the physical element K. Find also the Jacobian J of the mapping.

Solution:
The mapping is given by

$$x(\xi,\eta) = h_1\xi + (h_2 - h_1)\xi\eta, \qquad y(\xi,\eta) = k\eta,$$

and the Jacobian becomes

$$J(\xi,\eta) = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} h_1 + (h_2 - h_1)\eta & (h_2 - h_1)\xi \\ 0 & k \end{pmatrix}.$$

c) The mapping in **b)** is used to define the nodal basis functions $\varphi_\alpha(x,y) = \hat{\varphi}_{\hat{\alpha}}(\xi(x,y), \eta(x,y))$ on K. We would like to compute terms of the kind:

$$a^K_{\alpha,\beta} = \int_K \nabla\varphi_\alpha \cdot \nabla\varphi_\beta \, dxdy = \int_{\hat{K}} (?)\ d\xi d\eta.$$

Find an expression for the integrand (?) on the right hand side, in terms of J and the basis functions on $\hat{K}$.

Set $h_1 = 0.8$, $h_2 = 1.2$, $k = 0.8$ and find an approximation to $a^K_{\alpha,\alpha}$, where α refer to the node in the lower left corner, $(0,0)$. Use the simple numerical quadrature formula

$$\int_{\hat{K}} g(\xi,\eta) d\xi d\eta \approx g(\frac{1}{2}, \frac{1}{2})$$

to approximate the integral.

Solution:
We know from the lectures (show the details) that

$$\nabla\varphi_\alpha(x(\xi,\theta),y(\xi,\theta)) = J^{-T}\nabla\hat{\varphi}_{\hat{\alpha}}, \qquad \textit{and} \qquad dx\,dy = |J|d\xi d\eta.$$

so

$$a^K_{\alpha,\beta} = \int_{\hat{K}} (J^{-T}\nabla\hat{\varphi}_{\hat{\alpha}})\cdot(J^{-T}\nabla\hat{\varphi}_{\hat{\beta}})|J|d\xi d\eta.$$

Using the given values for h_1, h_2 and k gives

$$J = \begin{pmatrix} 0.8+0.4\eta & 0.4\xi \\ 0 & 0.8 \end{pmatrix}, \qquad |J| = 0.64+0.32\eta,$$

We also have that

$$\nabla\hat{\varphi}_{\hat{\alpha}} = \begin{pmatrix} -1+\xi \\ -1+\eta \end{pmatrix}$$

So the whole thing becomes quite nonlinear, but using the given quadrature formula, the whole thing becomes

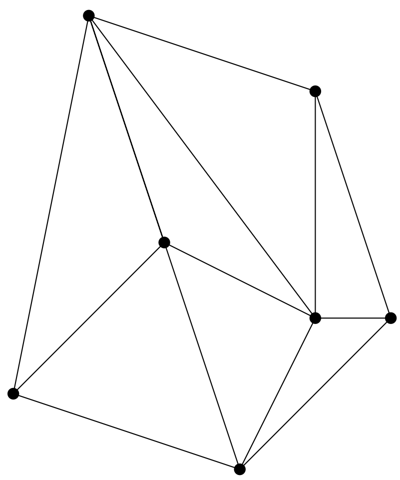
$$a^K_{\alpha,\alpha} \approx 0.4.$$

Problem 3

What is a Delaunay grid, and why is it attractive?

Is the grid to the right Delaunay? Justify your answer.

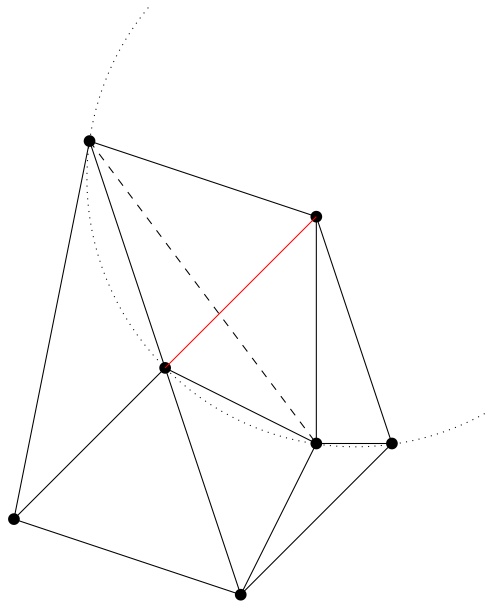
How can you change it to make it Delaunay?



Solution:

For the definition of the Delaunay grid, I refer to Quarteroni 147[a]
The attractive property is the second one, the max-min regularity property.
The interpolation error on one element K depends on the sphericity ρ_K of that element. The bigger ρ_k, the smaller error, see Theorem 4.4 in Quarteroni. And for a given set of gridpoints, the Delaunay triangulation optimize the grid with respect to this property.
The grid is obviously not Delaunay, since at least for one of the elements, the circumscribed circle contains a node, as indicated on the picture to the left.
The situation can be corrected by a diagonal exchange, the dashed line is replaced by the red one. By inspection, we can see that no other diagonal exchanges are necessary, thus the new grid is Delaunay.

[a] You are supposed to elaborate this somewhat more in your answer sheet.



Problem 4

Given the variational problem

$$\text{find } u \in V \text{ such that} \qquad a(u,v) = F(v) \qquad \forall v \in V \tag{3}$$

with

$$a(u,v) = \int_0^1 u_x v_x dx + \kappa \int_0^1 uv dx, \qquad F(v) = \int_0^1 v dx, \qquad V = H^1(0,1),$$

a) For which κ is there a unique solution to (3)? Justify your answer.

Solution:
For existence and uniqueness, use Lax-Milgram. Remember that

$$\|v\|^2_{H^1(0,1)} = \int_0^1 (v^2 + v_x^2) dx.$$

The Sobolev space $H^1(0,1)$ is a Hilbertspace.

The form F is clearly linear. It is also continuous, since

$$|F(v)| = |\int_0^1 v dx| = \|v\|_{L^2(0,1)} \leq \|v\|_{H^1(0,1)}.$$

The form $a(\cdot,\cdot)$ is obviously bilinear. It is continuous since

$$\begin{aligned}
|a(u,v)| &= |\int_0^1 u_x v_x dx + \kappa \int_0^1 uv dx| \\
&\leq \max(1,|\kappa|)\ |\int_0^1 (u_x v_x + uv) dx| \\
&\leq \max(1,|\kappa|)\ \|u\|_{H^1(0,1)} \|v\|_{H^1(0,1)} && \textit{Cauchy-Schwarz}
\end{aligned}$$

We also have to check for coercivity:

$$a(v, v) = \int_0^1 (v_x^2 + \kappa v^2)dx \geq \min(1, \kappa)\|v\|^2_{H^1(0,1)},$$

so a is coercive if $\kappa > 0$.

We conclude that the problem has a unique solution for $\kappa > 0$.

b) Let κ satisfy the conditions for solvability found in **a)**.

Assume that you want to find an approximation u_h to the solution by solving the variational problem on a finite dimensional subspace $V_h \subset H^1(0, 1)$. Prove that

$$\|u - u_h\|_{H^1(0,1)} \leq C\|u - v_h\|_{H^1(0,1)} \qquad \forall v_h \in V_h.$$

and find an appropriate constant C.

Solution:
See Quarteroni, p. 65. The constant is

$$C = \frac{M}{\alpha} = \frac{\max(1, \kappa)}{\min(1, \kappa)} = \max(\kappa, \frac{1}{\kappa}).$$

Appendix

Differential operators in polar coordinates (r, θ)

$$\operatorname{grad} g = \nabla g = \left(\frac{\partial g}{\partial r}, \frac{1}{r}\frac{\partial g}{\partial \theta}\right)^T$$

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}$$

$$\Delta g = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2}.$$