Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact during exam:
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Exam in TMA4220
Numerical Solution of Partial Differential Equations Using Element Methods

Wednesday December 5, 2012
Time: 15.00 – 19.00

Auxiliary materials: Simple calculator (Hewlett Packard HP30S or Citizen SR-270X)
All printed and hand written material.

Deadline for the grading: 21.12.2012.

Problem 1

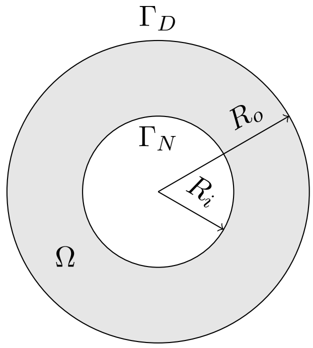
Given the Poisson equation

$$-\Delta u = f \quad \text{in} \quad \Omega \tag{1}$$

with boundary conditions

$$u = 0 \;\text{ on }\; \Gamma_D, \qquad \frac{\partial u}{\partial n} = q \;\text{ on }\; \Gamma_N.$$

where Ω is the domain between a circle of radius R_i and one of radius R_o, Γ_D is the outer boundary and Γ_N the inner.



a) Establish the weak formulation

$$\text{find } u \in V \text{ such that} \qquad a(u,v) = F(v), \qquad \forall v \in V. \tag{2}$$

That means, identify V and find expressions for a and F, In the following, let $f = -4$, $q = 0.5$, $R_i = 0.5$ and $R_o = 1$. Since f and q are both constant, we note that the solution u only depend on the distance from the center r, and the problem is reduced to a one-dimensional case.

b) Show that the bilinear form $a(u, v)$ in the weak formulation (2) now is

$$a(u,v) = \int_{0.5}^{1} r \frac{\partial u}{\partial r}\frac{\partial v}{\partial r}\, dr, \qquad V = \{v \in H^1(0.5, 1) \ : \ v(1) = 0\}.$$

Find also an expression for $F(v)$ in this case.

Hint: Use polar coordinates, see the appendix at the end of the set.

c) Find the exact solution $u(r)$ of this problem.

We will now like to find an approximation to the solution of the one-dimensional problem by use of the finite element method with linear, nodal basis functions on a uniform grid, that is $V_h = X_h^1$, and $h = 0.5/M$.

d) Find the elemental stiffness matrix and the elemental load vector for the element $K = [r_{k-1}, r_k]$, where $r_k = 0.5 + h \cdot k$, $k = 1, 2, \cdots, M$.

e) The finite element method can be formulated as

$$A_h \mathbf{u}_h = \mathbf{b}_h$$

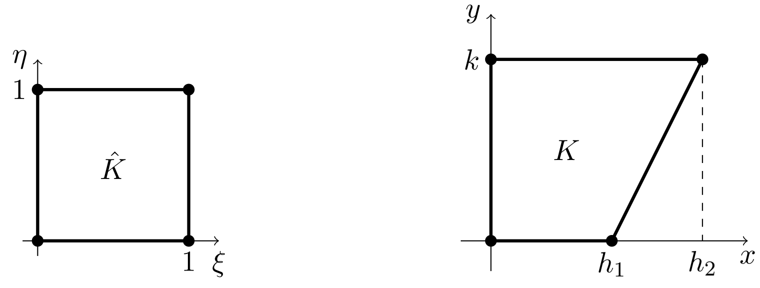
where $\mathbf{u}_h = (u_0, u_1, \cdots, u_{M-1})^T$ where $u_k \approx u(r_k)$. Show that A_h is the symmetric tridiagonal matrix

$$A_h = \begin{pmatrix} \gamma_0 & \beta_0 & & & \\ \beta_0 & \gamma_1 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{M-2} \\ & & & \beta_{M-2} & \gamma_{M-1} \end{pmatrix}$$

and find for β_k and γ_k (only for $k \neq 0$ and $k \neq M-1$.).

Problem 2

a) Given a quadratic reference finite element $\hat{K}$, with nodes in the corners. Write down the four bilinear nodal basis functions $\hat{\varphi}_{\hat{\alpha}}(\xi, \eta)$ for this element. The index $\hat{\alpha}$ refers to the nodes.



b) Find a bilinear mapping $x(\xi, \eta), y(\xi, \eta)$ mapping each node from the quadratic reference element to the corresponding node of the physical element K. Find also the Jacobian J of the mapping.

c) The mapping in **b)** is used to define the nodal basis functions $\varphi_\alpha(x, y) = \hat{\varphi}_{\hat{\alpha}}(\xi(x, y), \eta(x, y))$ on K. We would like to compute terms of the kind:

$$a^K_{\alpha,\beta} = \int_K \nabla\varphi_\alpha \cdot \nabla\varphi_\beta \, dxdy = \int_{\hat{K}} (?) \; d\xi d\eta.$$

Find an expression for the integrand (?) on the right hand side, in terms of J and the basis functions on $\hat{K}$.

Set $h_1 = 0.8$, $h_2 = 1.2$, $k = 0.8$ and find an approximation to $a^K_{\alpha,\alpha}$, where α refer to the node in the lower left corner, $(0, 0)$. Use the simple numerical quadrature formula

$$\int_{\hat{K}} g(\xi, \eta) d\xi d\eta \approx g(\frac{1}{2}, \frac{1}{2})$$

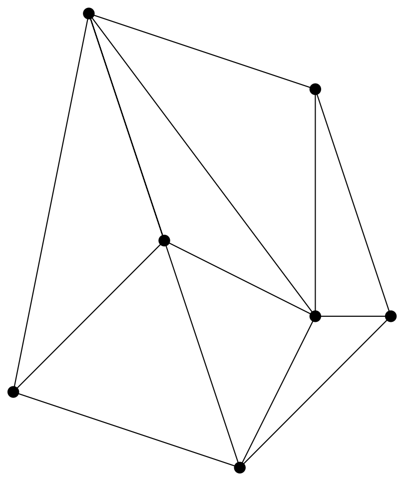
to approximate the integral.

Problem 3

What is a Delaunay grid, and why is it attractive?

Is the grid to the right Delaunay? Justify your answer.

How can you change it to make it Delaunay?



Problem 4
Given the variational problem

$$\text{find } u \in V \text{ such that} \qquad a(u,v) = F(v) \qquad \forall v \in V \tag{3}$$

with

$$a(u,v) = \int_0^1 u_x v_x dx + \kappa \int_0^1 uv dx, \qquad F(v) = \int_0^1 v dx, \qquad V = H^1(0,1),$$

a) For which κ is there a unique solution to (3)? Justify your answer.

b) Let κ satisfy the conditions for solvability found in **a)**.

Assume that you want to find an approximation u_h to the solution by solving the variational problem on a finite dimensional subspace $V_h \subset H^1(0,1)$. Prove that

$$\|u - u_h\|_{H^1(0,1)} \leq C\|u - v_h\|_{H^1(0,1)} \qquad \forall v_h \in V_h.$$

and find an appropriate constant C.

Appendix

Differential operators in polar coordinates (r, θ)

$$\begin{aligned}
\text{grad}\, g = \nabla g &= \left(\frac{\partial g}{\partial r}, \frac{1}{r}\frac{\partial g}{\partial \theta}\right)^T \\
\text{div}\mathbf{F} = \nabla \cdot \mathbf{F} &= \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} \\
\Delta g &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2}.
\end{aligned}$$