NTNU

Department of Mathematical Sciences

Examination paper for
TMA4220 Numerical Solution of partial differential equations

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Permitted examination support material:
Code C: Simple calculator (Hewlett Packard HP30S or Citizen SR-270X),
Written materials: A. Quarteroni, Numerical Models for Differential Problems,
Springer 2008, and S. Brenner and L. R. Scott,
The Mathematical Theory of Finite Element Methods, Springer 2005.

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Problem 1 Consider the two-dimensional Poisson equation

$$-\nabla^2 u = f \quad \text{in} \quad \Omega \tag{1}$$

with the boundary conditions

$$u = 0 \quad \text{on} \quad \partial\Omega_D \tag{2}$$
$$H(u) = \bar{t} \quad \text{on} \quad \partial\Omega_N. \tag{3}$$

where u is the unknown solution, f the applied loading, $\bar{t}$ are prescribed Neumann boundary conditions for the given Poisson problem defined on the polygonal domain $\Omega \in R^2$. The differential operator $H(u)$ is the so called "Neumann operator".

a) Use Galerkin's method and establish the weak formulation corresponding to the problem (1)-(3) on the form: Find $u \in X$ such that

$$a(u, v) = l(v) \qquad \forall v \in X \tag{4}$$

In particular, identify X, a and l for this problem. Identify also the expression for the "Neumann operator" $H(u)$ related to the Neumann boundary conditions.

b) Assume that we choose the finite element method to solve (4) numerically. Formulate the corresponding finite element variational formulation.

In the following assume homogeneous Dirichlet boundary conditions along the whole boundary $\partial\Omega$. Let the applied loading f be a constant defined on the whole domain Ω.

c) Compute the terms $a(\phi_9, \phi_9)$ and $l(\phi_9)$, where ϕ_9 is the basis function connected to node 9 in Figure 1a) and 1b), respectively.

d) Find $u_h \in X_h$, i.e. solve the finite element system for the problem displayed in Figure 1a) and b), respectively. Explain the difference in the finite element solution between Mesh-1 and Mesh-2.

e) Find $u_h \in X_h$, i.e. solve the finite element system for the problem displayed in Figure 2, i.e. for Mesh-3. Explain the difference in the finite element solution between Mesh-2 and Mesh-3.

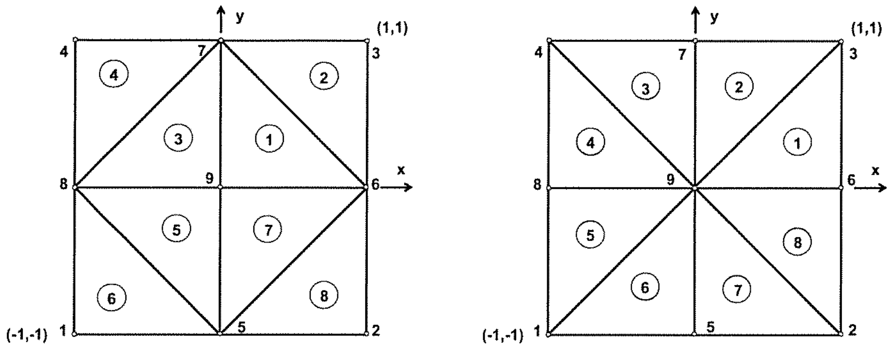

Figure 1: The domain $\Omega = [-1,-1] * [1,1]$ discretized with 8 linear triangles and 9 nodes. a) To the left we have Mesh-1 and b) to the right we have Mesh-2.

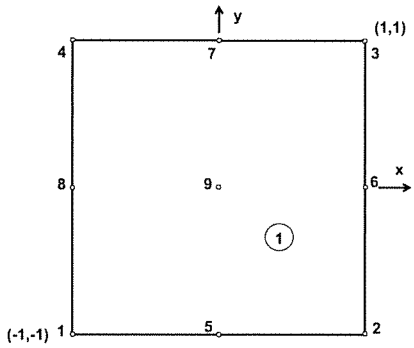

Figure 2: The domain $\Omega = [-1,-1] * [1,1]$ discretized with 1 biquadratic quadrilateral Lagrange element and 9 nodes (Mesh-3).

Problem 2

a) What characterize a finite element?

b) Define "affine coordinate mapping". Which property inherent in affine coordinate mappings is consider to be of special interest related to the finite element method.

c) Define the nodal basis functions for the 4 node "transfer element" shown in Figure 3, such that it is compatible with two neighboring standard 3 node linear Lagrange triangular element along the edge with the midpoint node, and compatible with a neighboring 3 node linear Lagrange triangular element along the two other edges.

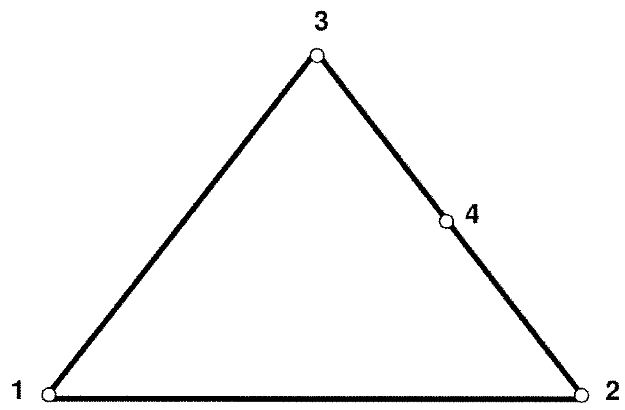

Figure 3: "Transfer element" with 4 nodes.

Problem 3

Consider the convection-diffusion problem:

$$-\kappa\, u_{xx} + U\, u_x = f \quad \text{in} \quad \Omega = (0,1) \ , \tag{5}$$

$$u(0) = 0 \ , \tag{6}$$

$$u(1) = 0 \ , \tag{7}$$

where κ is a constant, positive diffusion coefficient, and U is a constant convection velocity.

a) Derive a weak formulation of the problem (5)-(7) on the form: Find $u \in X$ such that

$$b(u, v) = l(v) \qquad \forall v \in X$$

In particular, identify X, b and l for this problem.

b) We are particularly interested in solving this problem numerically using the finite element method in the case of $\kappa = 0.05$, $U = 5$ and $f = 9$. Assume that we use a uniform grid with $K = 100$ linear elements. Will this be sufficient resolution in order to avoid oscillations in the discrete solution ?