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Examination paper for

TMA4220 Numerical Solution of Partial Differential Equations Using Element Methods

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Problem 1

a) Consider the equation

$$b \cdot \nabla u + \sigma u = \mu \Delta u + f \qquad \text{for } x \in \Omega \tag{1}$$

where b is a constant vector, σ and μ are positive constants, and $f \in L^2(\Omega)$. Derive the weak formulation of (1) with Neumann boundary conditions

$$\frac{\partial u}{\partial n} = h \qquad \text{on } \partial\Omega$$

for some continuous function $h : \partial\Omega \to \mathbb{R}$.

b) Consider the equation (1), but with

$$b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \sigma = 0, \quad \mu = 0, \quad f(x) \equiv 2 \quad \text{and} \quad \Omega = (0,1) \times (0,1).$$

If we want a homogeneous Dirichlet (i.e., $u = 0$) boundary condition, on which part of $\partial\Omega$ should this be imposed? Write down the resulting boundary-value problem (PDE + boundary values) and find the exact solution.

c) Consider the values from **b)**, but with $\sigma = 1$ and $\mu = \frac{1}{100}$. We discretize (1) using the finite element method with X_h^1 elements. How small must h be in order to get an oscillation-free numerical solution? (Here you do *not* have to justify your answer.)

Problem 2 Consider the time-dependent problem

$$\begin{cases} \frac{\partial u}{\partial t} = bu + c\Delta u & \text{for } x \in \Omega, \ t > 0 \\ u(x,0) = \bar{u}(x) & \text{for } x \in \Omega \\ u(x,t) = 0 & \text{for } x \in \partial\Omega, t > 0. \end{cases} \tag{2}$$

a) Write down the weak formulation of (2), that is, a statement of the form

$$\text{find } u(t) \in V \text{ such that } \ldots \text{ for all } v \in V.$$

Be sure to specify the function space V.

b) For what values of $b, c \in \mathbb{R}$ is (2) parabolic?

c) We discretize (2) using the finite element method and Backward Euler time stepping. What is the vector equation satisfied by the vector of degrees of freedom $\xi^n \in \mathbb{R}^N$? (Here, $n = 0, 1, 2, \ldots$ is the time step and N is the number of degrees of freedom.)

Problem 3 Consider the problem

$$\begin{cases} -u'' = f & \text{for } x \in (0,1) \\ u(0) = 0, \;\; u(1) = 1. \end{cases} \tag{3}$$

a) Derive the finite element method for (3) using piecewise linear elements on the uniform mesh

$$K_i = [x_{i-1}, x_i], \qquad i = 1, \dots, 5$$

where $x_i = \frac{i}{5}$ for $i = 0, \dots, 5$. Compute the components of the stiffness matrix A.

b) Assume that you have computed the numerical solution u_h from **a)** with $f(x) = x(2x-1)$.

(i) State the basic error estimate for this problem in terms of f.

(ii) In which cell K_i would you expect the largest error? If you could add a node x_6 to the mesh, where should you place it in order to decrease the overall error by as much as possible?

Problem 4 Consider the Galerkin problem

$$\text{find } u \in V \text{ such that } a(u,v) = F(v) \text{ for all } v \in V$$

where a, F and V satisfy the conditions of the Lax–Milgram theorem. We approximate the solution u with the finite element method, using a finite element space $V_h \subset V$. Prove that the approximate solution u_h satisfies

$$\|u - u_h\|_V \leqslant C\|u - v_h\|_V \qquad \text{for all } v_h \in V_h$$

for some $C > 0$. What is this result called?

Problem 5 Let $K_i = [x_{i-1}, x_i]$ (for $i = 1, \dots, N$) be a partition of $\Omega = (0,1)$. Let V_h be the space of functions which are

- continuously differentiable on *all* of Ω,
- piecewise r-th order polynomials on $\{K_1, \dots, K_N\}$,

that is,

$$V_h = \Big\{ v \in C^1(\Omega) \;:\; v\Big|_{K_i} \in \mathcal{P}^r, \; i = 1, \dots, N \Big\}.$$

What is the smallest $r \in \{0, 1, 2, \dots\}$ which we can use in this definition? For this value of r, where would you put the degrees of freedom in each element domain K_i?

Problem 6

a) Consider the map Φ which maps K to $\hat{K}$ in Figure 1. (Here, a, b and c are given, positive numbers.) Write down Ψ and Ψ^{-1}, as well as the Jacobian J of Ψ^{-1}.

b) Assume that we have defined some set of shape functions $\hat{\varphi}_\alpha : \hat{K} \to \mathbb{R}$ (for $\alpha = 0, \dots, r$) on the reference triangle $\hat{K}$. We map them to K by setting $\varphi_\alpha(x) = \hat{\varphi}_\alpha(\Psi(x))$, and would like to compute the integral

$$A^K_{\alpha,\beta} = \int_K \nabla\varphi_\alpha(x) \cdot \nabla\varphi_\beta(x)\ dx.$$

Write this integral as an integral over $\hat{K}$ using the reference shape functions $\hat{\varphi}_\alpha$, $\hat{\varphi}_\beta$ and the Jacobian J.

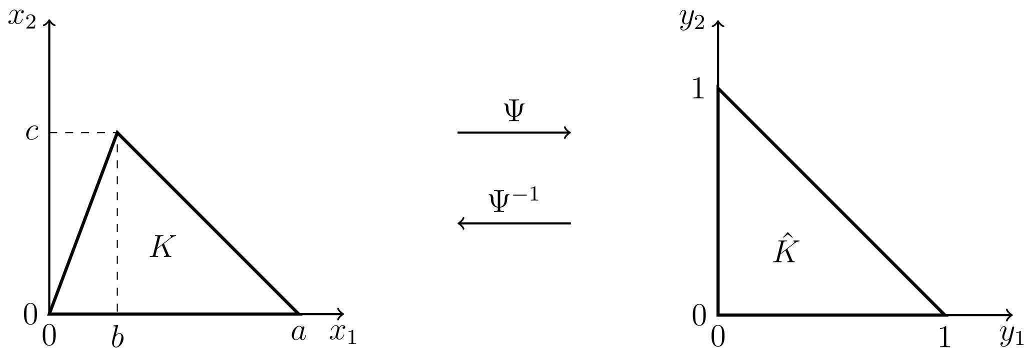

Figure 1: $\Psi : K \to \hat{K}$ maps the triangle K to the reference triangle $\hat{K}$.

Problem 7

a) Define what we mean by the Delaunay triangulation of N points $x_1, \dots, x_N \in \mathbb{R}^2$.

b) Draw the Delaunay triangulation of the seven points in Figure 2. (You can either draw on the exam sheet or copy the figure to a separate sheet.)

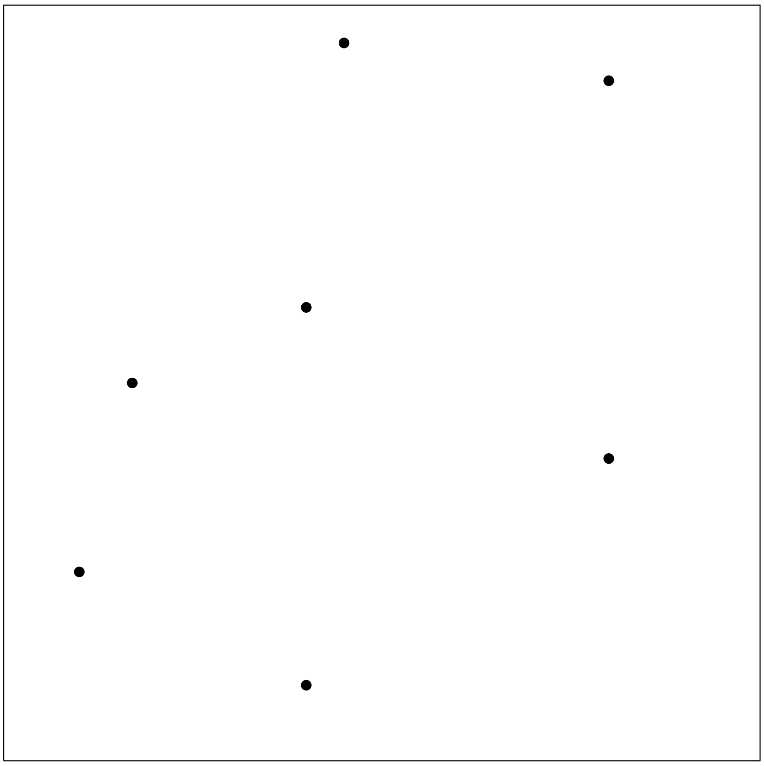

Figure 2: Delaunay triangulate these seven points!