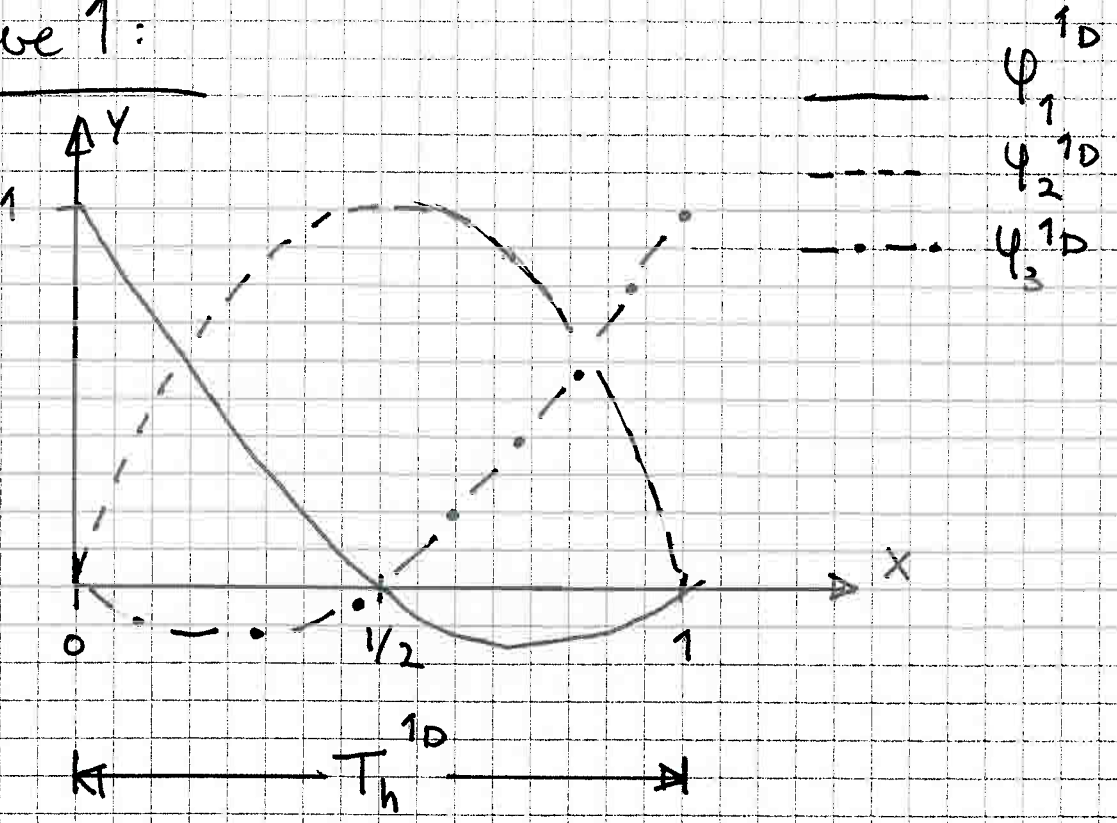
TMA 4220 , EKSAMEN VÅR 2006

LØYSNINGSFORSLAG.

Oppgåve 1:

a)



$$\varphi_1^{1D}(x) = 2\left(x-\tfrac{1}{2}\right)(x-1)$$

$$\varphi_2^{1D}(x) = -4(x-0)(x-1) = \underline{\underline{-4x(x-1)}}$$

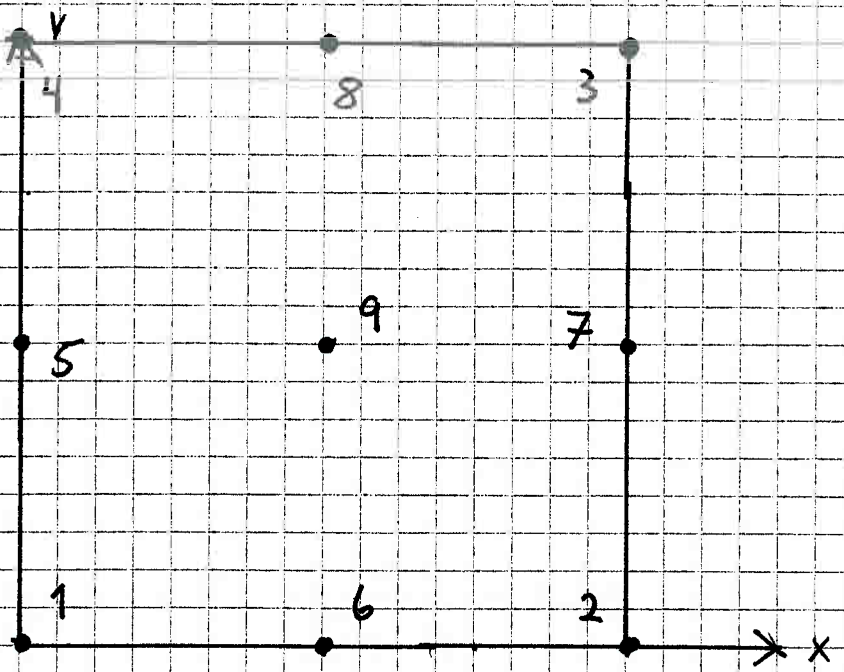
$$\varphi_3^{1D}(x) = 2\left(x-\tfrac{1}{2}\right)(x-0) = \underline{\underline{2x\left(x-\tfrac{1}{2}\right)}}$$

Legg merke til at

$$\varphi_i^{1D}(x_j) = \delta_{ij}\,, \quad i,j = 1,2,3$$

$$x_1 = 0\,, \quad x_2 = \frac{1}{2}\,, \quad x_3 = 1$$

b)



$$\varphi_1^{2D}(x,y) = \varphi_1^{1D}(x)\ \varphi_1^{1D}(y)$$

$$\varphi_2^{2D}(x,y) = \varphi_3^{1D}(x)\ \varphi_1^{1D}(y)$$

$$\varphi_3^{2D}(x,y) = \varphi_3^{1D}(x)\ \varphi_3^{1D}(y)$$

$$\varphi_4^{2D}(x,y) = \varphi_1^{1D}(x)\ \varphi_3^{1D}(y)$$

$$\varphi_5^{2D}(x,y) = \varphi_1^{1D}(x)\ \varphi_2^{1D}(y)$$

$$\varphi_6^{2D}(x,y) = \varphi_2^{1D}(x)\ \varphi_1^{1D}(y)$$

$$\varphi_7^{2D}(x,y) = \varphi_3^{1D}(x)\ \varphi_2^{1D}(y)$$

$$\varphi_8^{2D}(x,y) = \varphi_2^{1D}(x)\ \varphi_3^{1D}(y)$$

$$\varphi_9^{2D}(x,y) = \varphi_2^{1D}(x)\ \varphi_2^{1D}(y)$$

c)

$$-\int_{\Omega} (\nabla^2 u)\, v \, d\Omega = \int_{\Omega} v \, d\Omega$$

$$\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega - \int_{\partial\Omega} \cancel{v \frac{\partial u}{\partial n}} \, d\Gamma = \int_{\Omega} v \, d\Omega$$

$$v\big|_{\partial\Omega} = 0$$

Finn $u \in X$ slik at

$$\underbrace{\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega}_{a(u,v)} = \underbrace{\int_{\Omega} v \, d\Omega}_{l(v)}, \quad \forall\, v \in X$$

der $X = H^1_0(\Omega) = \left\{ w \in H^1(\Omega),\ w\big|_{\partial\Omega} = 0 \right\}$

d) Systemet har 1 fridomsgrad kopla i node 9.

e) Pga $a(\cdot,\cdot)$ er symmetrisk

vil $\left(A_h^{T_h^{2D}}\right)_{5,3} = \left(A_h^{T_h^{2D}}\right)_{3,5} = \underline{\underline{\frac{1}{9}}}$

$\left(A_h^{T_h^{2D}}\right)_{1,1}$:

La $\sigma = 1 - x$ då er

$$\varphi_1^{1D}(x) = \varphi_1^{1D}(1-\cancel{x}) = 2\left(1-\sigma-\tfrac{1}{2}\right)(1-\sigma-1)$$

$$= 2\sigma\left(\sigma-\tfrac{1}{2}\right) = \varphi_3^{1D}(\sigma)$$

$$\frac{d}{dx}\varphi_1^{1D}(x) = \frac{d}{d\sigma}\varphi_3^{1D}(\sigma)\cdot\frac{d\sigma}{dx}$$

$$= \varphi_3^{1D\,\prime}(\cancel{\sigma})(-1)$$

$$\underline{= -\varphi_3^{1D\,\prime}(\sigma)}$$

$$\left(A_h^{T_h^{2D}}\right)_{1,1} = \int_\Omega \nabla \varphi_1^{2D} \cdot \nabla \varphi_1^{2D} \, d\Omega$$

$$= \int_0^1 \int_0^1 \left[\left(\frac{\partial \varphi_1^{2D}}{\partial x}\right)^2 + \left(\frac{\partial \varphi_1^{2D}}{\partial y}\right)^2 \right] dx\,dy$$

$$= \int_0^1 \int_0^1 \left[\left(\frac{d\varphi_1^{1D}(x)}{dx}\right)^2 \left(\varphi_1^{1D}(y)\right)^2 + \left(\varphi_1^{1D}(x)\right)^2 \left(\frac{d\varphi_1^{1D}}{dy}(y)\right)^2 \right] dx\,dy$$

$$= \int_0^1 \left(\varphi_1^{1D}(y)\right)^2 dy \cdot \int_0^1 \left(\frac{d\varphi_1^{1D}}{dx}(x)\right)^2 dx$$

$$+ \int_0^1 \left(\varphi_1^{1D}(x)\right)^2 dx \cdot \int_0^1 \left(\frac{d\varphi_1^{1D}}{dy}(y)\right)^2 dy$$

$$= 2 \cdot \int_0^1 \left(\varphi_1^{1D}(y)\right)^2 \cdot \int_0^1 \left(\frac{d\varphi_1^{1D}}{dx}(x)\right)^2 dx$$

Vi innfører $\sigma = 1 - x$, $\eta = 1 - y$
$d\sigma = -dx$, $d\eta = -dy$

$$\left(A_h^{T_h^{2D}}\right)_{1,1} = 2\int_1^0 \left[\varphi_3^{1D}(\eta)\right]^2 (-d\eta)$$

$$\cdot \int_1^0 \left[-\frac{d}{d\sigma}\varphi_3^{1D}(\sigma)\right]^2 (-d\sigma)$$

$$= 2\int_0^1 \left[\varphi_3^{1D}(\eta)\right]^2 d\eta \cdot \int_0^1 \left[\frac{d}{d\sigma}\varphi_3^{1D}(\sigma)\right]^2 d\sigma$$

$$= \int_\Omega \nabla\varphi_3^{2D} \cdot \nabla\varphi_3^{2D}\, d\sigma$$

$$= \left(A_h^{T_h^{2D}}\right)_{3,3} = \underline{\underline{\frac{28}{45}}}$$

Kortversjoner er bette å veive med henda og hevde:

$$\left(A_h^{T_h^{2D}}\right)_{j,j} = \frac{28}{45} \quad , \quad j = 1,2,3,4$$

f) $\left(A_h^{T_h^{2D}}\right)_{9,9} u_9 = (F_h)_9 = \frac{4}{9}$

$$\frac{256}{45} u_9 = \frac{4}{9}$$

$$u_9 = \frac{4}{9} \frac{45}{256}$$

$$u_9 = \frac{5}{64}$$

$$\underline{\underline{u_h(x,y) = \frac{5}{64} \varphi_9^{2D}(x,y)}}$$

Oppgåve 2:

a) PDE, sterkt form:

$$-u_{xx} + u_x + u = f, \quad x \in \Omega = (0,1)$$

$$u'(0) = u'(1) = 0$$

Utleiing svak form:

$$-\int_\Omega u_{xx} v \, dx + \int_\Omega u_x v \, dx + \int_\Omega u v \, dx = \int_\Omega f v \, dx,$$

$$\forall v \in X$$

$$\int_\Omega u_x v_x \, dx - \underbrace{\left[u_x v\right]_0^1}_{u_x(0) = u_x(1) = 0} + \int_\Omega u_x v \, dx + \int_\Omega u v \, dx = \int_\Omega f v \, dx,$$

$$\forall v \in X$$

Svak formulering: Finn $u \in X$ slik at

$$\int_\Omega u_x v_x \, dx + \int_\Omega u_x v \, dx + \int_\Omega u v \, dx = \int_\Omega f v \, dx,$$

$$\forall v \in X$$

der $X = H^1(\Omega)$.

og

$$a(u,v) = \int_\Omega (u_x v_x + u_x v + u v)\, dx$$

$$l(v) = \int_\Omega f v\, dx$$

b)

$$a(u,v) = (u,v)_{L^2(\Omega)} + (u_x, v_x)_{L^2(\Omega)} +$$

$$+ (u_x, v)_{L^2(\Omega)}$$

$$= (u,v)_{H^1(\Omega)} + (u_x, v)_{L^2(\Omega)}$$

$$|a(u,v)| = \left| (u,v)_{H^1(\Omega)} + (u_x, v)_{L^2(\Omega)} \right|$$

TREKANT ULIKSKAP: $\leq \left|(u,v)_{H^1(\Omega)}\right| + \left|(u_x,v)_{L^2(\Omega)}\right|$

CAUCHY-SCHARZ: $\leq \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} + \|u_x\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$

$$\underline{\underline{\leq 2 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}}}$$

$$\underline{\underline{C = 2}}$$

$$a(v,v) = \int_\Omega \left(v_x^2 + v_x v + v^2\right) dx$$

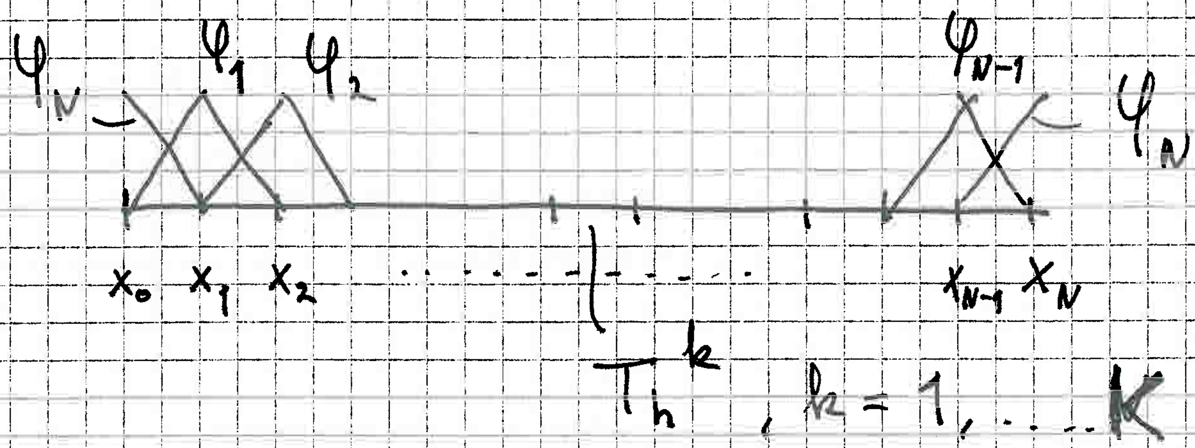
$$= \frac{1}{2}\int_\Omega \left(v_x^2 + v^2\right) dx + \underbrace{\frac{1}{2}\int_\Omega \left(v_x + v\right)^2 dx}_{\geq 0}$$

$$\geq \frac{1}{2} \|v\|_{H^1}^2$$

$$\alpha = \frac{1}{2}$$

Oppgåve 3:

a) K lineære element:



For K lineære element er $N = K$. Det er totalt $N+1$ nodar, men node 0 og N er assosiert med same fridomsgrad:

$$\dim \bar{X}_h = N = K$$

$$\bar{X}_h = \text{span}\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$$

La $u_h(x) = \sum_{j=1}^{N} u_{h_j}(t)\,\varphi_j(x)$

og la $v = \varphi_i$, $i = 1, \dots, N$ og

set inn i likning (4):

$$\frac{d}{dt}(u, v) + c(u, v) = 0 \quad \forall v \in X$$

Dette gir systemet

$$\frac{d}{dt}\underline{M}_h \underline{u}_h + \underline{C}_h \underline{u}_h = 0$$

Der $\underline{M}_h$ er massematrisa

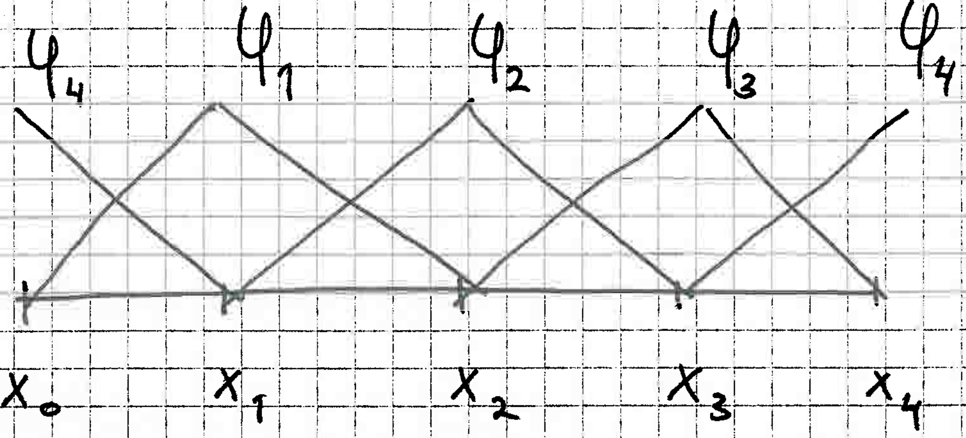
$$(\underline{M}_h)_{i,j} = (\varphi_j, \varphi_i)$$

~~og~~ $\underline{C}_h$: $(\underline{C}_h)_{i,j} = c(\varphi_j, \varphi_i)$

og $\underline{u}_h(t) = [u_{h_1}(t), u_{h_2}(t) \dots u_{h_N}(t)]^T$

b) $K = 4$:

ALTERNATIV 1:



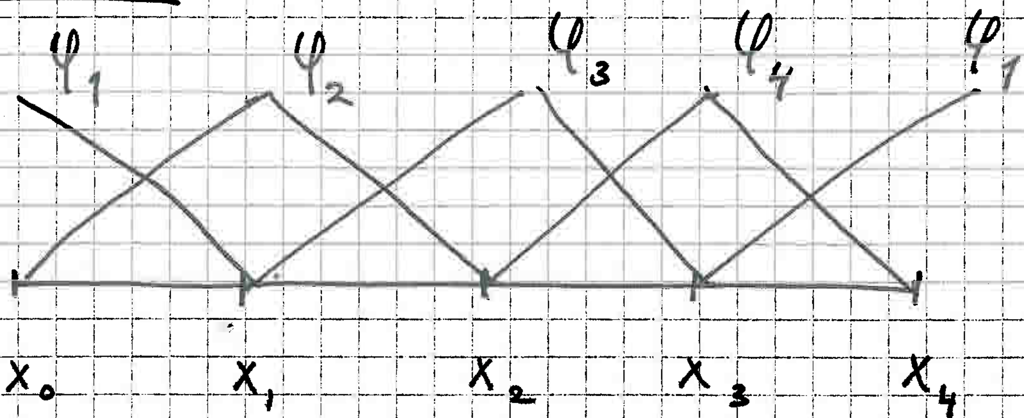
LOKAL – TIL – GLOBAL:

α \ k	1	2	3	4
1	0	1	2	3
2	1	2	3	4

NODE – TIL – FRIHETSGRAD

node	0	1	2	3	4
frihetsgrad	4	1	2	3	4

ALTERNATIV 2:



NODE-TIL-FRIHETSGRAD

NODE	0	1	2	3	4
FRIHETSGRAD	1	2	3	4	1

c) ALT. 1:

$$\underline{A}_h = \frac{V}{2} \begin{array}{cccc|c} 1 & 2 & 3 & 4 & \\ \left[\begin{array}{cccc} -1+1 & 1 & & -1 \\ -1 & 1-1 & 1 & \\ & -1 & 1-1 & 1 \\ +1 & & -1 & 1-1 \end{array}\right] & & & & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

For T_h^1:

Lokal node	1	2
FRIHETSGRAD	4	1

$$\underline{A}_h = \frac{V}{2} \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

ALT. 2:

$$\underline{A}_h = \frac{U}{2}\begin{bmatrix} -1+1 & 1 & & -1 \\ -1 & 1-1 & 1 & \\ & -1 & 1-1 & 1 \\ +1 & & -1 & 1-1 \end{bmatrix}$$

FOR T_h^4

LOKALNODE	1	2
FRIHETSGRAD	4	1

$$\underline{A}_h = \frac{U}{2}\begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

d) SJÅ NOTAT OM

TIDSAVHENGIG KONVEKSJONSPROBLEM.