1. Find all extreme subsets (faces) and extreme points of the following sets in the plane:
a) $\{(x, y) \in \mathbb{R}^2 : \ x^2 + y^2 \leq 1\}$,
b) $\{(x, y) \in \mathbb{R}^2 : \ |x| + |y| \leq 1\}$,
c) $\{(x, y) \in \mathbb{R}^2 : \ x \leq 0\}$,
d) $\{(x, y) \in \mathbb{R}^2 : \ x = y\}$.
Then find the cardinality of the set of extreme subsets in each case.

2. Prove that a point a in a convex set K is an extreme point of K if and only if $K \setminus \{a\}$ is a convex set.

3. Let P be the set of all strictly positive functions on the unit interval. Prove that P is a convex set without extreme points.

4. Find a compact convex set K in $\mathbb{R}^3$ for which the set of extreme points is not closed. (*Hint*: Put $A = \{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 \leq 1, \ z = 0\} \cup \{(0, -1, 1), (0, -1, -1)\})$, and put $K = \mathrm{co}(A)$.)

5. Prove that the set of maximizers of a convex function (resp. minimizers of a concave function) defined on a convex set is either empty or an extreme set.

6. (Bauer Maximum Principle) Let K be a compact convex subset of a normed space. Show that every continuous linear functional achieves its maximum and minimum values on K at extreme points of K. (*Hint*: Use Problem 5 and the Krein-Milman theorem.)

7. Let c_0 denote the space of all sequences converging to zero with the norm $\|x\| = \sup_i |\xi_i|$, for $x = (\xi_1, \xi_2, \cdots) \in c_0$. Prove that the closed unit ball of c_0 has no extreme points, but this does not contradict the Krein-Milman theorem.