Norges teknisk–
naturvitenskapelige universitet
Institutt for matematiske fag

Bokmål

LØSNINGSFORSLAG TIL EKSAMENSOPPGAVENE I

EMNE TMA4245 STATISTIKK
xx. august 2011
Tid: 09:00–13:00

Oppgave 1

a)

b) A og B er uavhengige hvis og bare hvis $P(A \cap B) = P(A)\,P(B)$. Ser på

$$\begin{aligned} P(A' \cap B') &= P((A \cup B)') = 1 - P(A \cup B) \\ &= 1 - P(A) - P(B) + P(A \cap B) \\ &= 1 - P(A) - P(B) + P(A)\,P(B) \\ &= (1 - P(A))(1 - P(B)) = P(A')\,P(B') \end{aligned}$$

Dermed er A' og B' uavhengige.

$$\begin{aligned} P(A) &= P(A \cap (B \cup B')) = P((A \cap B) \cup (A \cap B')) \\ &= P(A \cap B) + P(A \cap B') = P(A)\,P(B) + P(A \cap B') \end{aligned}$$

Det gir at

$$P(A \cap B') = P(A) - P(A)\,P(B) = P(A)(1 - P(B)) = P(A)P(B'))$$

Dermed er A og B' uavhengige. Helt tilsvarende bevis for at A' og B er uavhengige.

Oppgave 2

a) (i)

$$P\big(X > 2000\big) = P\big(\frac{X-1500}{400} > \frac{2000-1500}{400}\big) = \Phi\big(-500/400\big) = \Phi(-1.25) = 0.1056$$

(ii) $X + Y \sim N(3000, \overset{2\cdot 400^2}{\cancel{400^2}})$. Det gir at $P(X + Y > 3000) = 0.5$.

(iii) $X - 2Y \sim N(-1500, 5 \cdot 400^2)$. Dermed

$$P\big(X - 2Y > 0\big) = P\big(\frac{X-2Y+1500}{400\sqrt{5}} > \frac{1500}{400\sqrt{5}}\big) = \Phi\big(-1.68\big) = 0.0465$$

b)

Oppgave 3

a)

$$\begin{aligned} P(Z > 10) &= 1 - P(Z \leq 10) = 1 - F(10; 0.05) \\ &= 1 - (1 - e^{-0.05\cdot 10}) = e^{-0.5} = 0.607 \end{aligned}$$

I den neste deloppgaven benytter vi egenskapen at eksponensialfordelingen er "uten hukommelse" og får

$$P(Z > 20 | Z > 10) = P(Z > 10) = 0.607$$

b)

$$\begin{aligned} P(M = m) &= P(m \leq Z < m + 1) \\ &= P(Z \leq m + 1) - P(Z \leq m) \\ &= F(m + 1; \lambda) - F(m; \lambda) \\ &= \big(1 - e^{-\lambda(m+1)}\big) - \big(1 - e^{-\lambda m}\big) \\ &= e^{-\lambda m} - e^{-\lambda(m+1)} \\ &= (1 - e^{-\lambda})e^{(-\lambda m)} \end{aligned}$$

Oppgave 4

a) $T \sim \text{eksp}(\frac{z}{\mu}) \quad \text{E}(T) = \frac{\mu}{z}$

$\mu = 1000, \;\; z = 2.0$

$P(T \leq 1000) = \int_0^{1000} \frac{z}{\mu} e^{-\frac{z}{\mu}x} dx = \int_0^{1000} \frac{1}{500} e^{-\frac{x}{500}} dx = [-e^{-\frac{x}{500}}]_0^{1000} = 1 - e^{-2} = 0.86$

$P(T \leq 1000) = 0.5 \quad \Leftrightarrow \quad 1 - e^{-\frac{1000z}{1000}} = 0.5$

$e^{-z} = 0.5 \quad \Leftrightarrow \quad z = -\ln 0.5 = 0.69$

$z_1 = 1.0, \quad z_2 = 2.0$

$P(T_2 \geq T_1) = ?$

Finner simultanfordelingen til T_1 og T_2:

$f(t_1, t_2) = \frac{z_1}{\mu} e^{-\frac{z_1}{\mu}t_1} \frac{z_2}{\mu} e^{-\frac{z_2}{\mu}t_2}$ siden T_1 og T_2 er uavhengige.

$$
\begin{aligned}
P(T_2 \geq T_1) &= \int_0^\infty \int_{t_1}^\infty f(t_1, t_2) dt_2 dt_1 = \frac{z_1 z_2}{\mu^2} \int_0^\infty \int_{t_1}^\infty e^{-\frac{z_1}{\mu}t_1} e^{-\frac{z_2}{\mu}t_2} dt_2 dt_1 \\
&= \frac{z_1 z_2}{\mu^2} \int_0^\infty [-\frac{\mu}{z_2} e^{-\frac{z_1}{\mu}t_1 - \frac{z_2}{\mu}t_2}]_{t_1}^\infty dt_1 = \frac{z_1 z_2}{\mu^2} \frac{\mu}{z_2} \int_0^\infty e^{-\frac{z_1}{\mu}t_1 - \frac{z_2}{\mu}t_2} dt_1 \\
&= \frac{z_1}{\mu} [-\frac{\mu}{z_1+z_2} e^{-(\frac{z_1+z_2}{\mu})t_1}]_0^\infty = \frac{z_1}{z_1+z_2} = \frac{1.0}{1.0+2.0} = \frac{1}{3}
\end{aligned}
$$

b) SME for μ:

$f(t_1, \ldots, t_n; \mu, z_1, \ldots, z_n) = \prod_{i=1}^n \frac{z_i}{\mu} e^{-\frac{z_i}{\mu}t_i}$

$L(\mu; t_1, \ldots, t_n, z_1, \ldots, z_n) = \prod_{i=1}^n \frac{z_i}{\mu} e^{-\frac{z_i}{\mu}t_i}$

$l(\mu) = \ln L(\mu) = \sum_{i=1}^n \ln z_i - n \ln \mu - \sum_{i=1}^n \frac{z_i}{\mu} t_i$

$\frac{\partial l}{\partial \mu} = -\frac{n}{\mu} + \sum_{i=1}^n \frac{z_i t_i}{\mu^2} = 0$

$n = \sum_{i=1}^n \frac{z_i t_i}{\mu}$

$\mu = \frac{1}{n} \sum_{i=1}^n z_i t_i$ Dermed er SME $\widehat{\mu} = \frac{1}{n} \sum_{i=1}^n z_i T_i$.

$$
\text{E}(\widehat{\mu}) = \text{E}(\frac{1}{n} \sum_{i=1}^n z_i T_i) = \frac{1}{n} \sum_{i=1}^n z_i \text{E}(T_i) = \frac{1}{n} \sum_{i=1}^n z_i \frac{\mu}{z_i} = \frac{1}{n} \sum_{i=1}^n \mu = \mu
$$

Dvs. estimatoren er forventningsrett.

$$\begin{aligned}\mathrm{Var}(\hat{\mu}) &= \mathrm{Var}(\tfrac{1}{n}\sum_{i=1}^{n} z_i T_i) = \tfrac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(z_i T_i) = \tfrac{1}{n^2}\sum_{i=1}^{n} z_i^2 \mathrm{Var}(T_i)\\ &= \tfrac{1}{n^2}\sum_{i=1}^{n} z_i^2 \tfrac{\mu^2}{z_i^2} = \tfrac{1}{n^2}\sum_{i=1}^{n}\mu^2 = \tfrac{\mu^2}{n}\end{aligned}$$

c) MGF for T_i: $M_{T_i}(t) = \frac{\frac{z_i}{\mu}}{\frac{z_i}{\mu}-t}$ (Funnet i tabell.)

$V = \frac{2n\widehat{\mu}}{\mu} = \frac{2\sum_{i=1}^{n} z_i T_i}{\mu} = \sum_{i=1}^{n} \frac{2z_i}{\mu} T_i$

$M_{\frac{2z_i}{\mu}T_i}(t) = \frac{\frac{z_i}{\mu}}{\frac{z_i}{\mu} - \frac{2z_i}{\mu}t} = (1-2t)^{-1}$ (Bruker at $M_{aX}(t) = M_X(at)$)

$M_V(t) = \prod_{i=1}^{n}(1-2t)^{-1} = (1-2t)^{-n}$

(Bruker at $M_{\sum_{i=1}^{n} X_i}(t) = \prod_{i=1}^{n} M_{X_i}(t)$)

$(1-2t)^{-n}$ er MGF for kji-kvadratfordelingen med $2n$ frihetsgrader. V har samme MGF som kji-kvadratfordelingen med $2n$ frihetsgrader, derfor er $V \sim \chi^2_{2n}$.

d) $(1-\alpha)100\%$ konfidensintervall for μ:

Bruker at $V = \frac{2n\widehat{\mu}}{\mu} \sim \chi^2_{2n}$.

$$\begin{aligned}&P(z_{1-\alpha/2,2n} \le V \le z_{\alpha/2,2n}) = 1-\alpha\\ &P(z_{1-\alpha/2,2n} \le \tfrac{2n\widehat{\mu}}{\mu} \le z_{\alpha/2,2n}) = 1-\alpha\\ &P(\frac{z_{1-\alpha/2,2n}}{2n\widehat{\mu}} \le \tfrac{1}{\mu} \le \frac{z_{\alpha/2,2n}}{2n\widehat{\mu}} \le \tfrac{1}{\mu}) = 1-\alpha\\ &P(\frac{2n\widehat{\mu}}{z_{\alpha/2,2n}} \le \mu \le \frac{2n\widehat{\mu}}{z_{1-\alpha/2,2n}}) = 1-\alpha\end{aligned}$$

Det gir konfidensintervallet $[\frac{2n\widehat{\mu}}{z_{\alpha/2,2n}}, \frac{2n\widehat{\mu}}{z_{1-\alpha/2,2n}}]$

$\alpha = 0.10$, $n = 10$, $\widehat{\mu} = 1270.38$

$z_{1-\alpha/2,2n} = z_{0.95,20} = 10.85$, $z_{\alpha/2,2n} = z_{0.05,20} = 31.41$

Innsatt disse tallverdiene blir konfidensintervallet $[808.90, 2341.71]$